Correlation spectrum for expanding maps and its relation to dynamical sensitivity

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Abstract

For a class of one-dimensional expanding maps we relate the two fundamental notions in the theory of dynamical systems: sensitivity to initial conditions (quantified by Lyapunov exponent or entropy) and mixing (measured via decay of correlation functions). More precisely, for piecewise linear expanding Markov maps on the interval observed via piecewise analytic functions, we show that the Lyapunov exponent Λ provides a barrier to the exponential rate of mixing, by establishing a lower bound on the subleading eigenvalue λ_2 of the transfer operator via $|\lambda_2| \geq e^{-2\Lambda}$.

Motivated by the question whether a similar bound in terms of the Lyapunov exponent can be obtained in the nonlinear setting, we construct a family of expanding maps for which the entire spectrum of the associated transfer operator is explicitly known. In particular, for any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ we construct an analytic expanding circle map such that the eigenvalues of the associated transfer operator (acting on holomorphic functions) are precisely the nonnegative powers of λ and $\overline{\lambda}$. Considered on the interval, these maps provide counterexamples to an old conjecture on the reality of spectra.

These examples belong to a special class of circle maps arising from finite Blaschke products. Their analytic features allow us to determine the entire spectrum of the associated transfer operators (on spaces of holomorphic functions) in terms of multipliers of attracting fixed points. This is achieved by deriving a natural representation of the respective adjoint operators in terms of certain composition operators. Using this explicit spectral information we then obtain examples of nonlinear expanding interval maps with arbitrarily fast exponential mixing but bounded Lyapunov exponent.

Contents

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Introduction

Chaos is the apparent randomness of a deterministic time-evolving system giving rise to motion too complicated to predict reliably. Measuring the strength of chaos amounts to quantifying the rate at which our uncertainty of prediction grows as we expand the time horizon.

The theory of dynamical systems provides a mathematical framework to study the behaviour of systems evolving in time. From an abstract perspective, a deterministic dynamical system is a set of states, usually referred to as the phase space of the system, equipped with an evolution law, describing the temporal transitions between these states. Even a simple evolution law can generate rather complicated behaviour, and make the system appear chaotic. Although there is no universally accepted mathematical definition of chaos, this notion is commonly used to describe deterministic systems demonstrating behaviour similar to purely random systems.

The source of chaotic motion is often attributed to *sensitive dependence on initial conditions*. Any uncertainty in the initial conditions risks being (exponentially) magnified, so that two initially arbitrarily close points will typically head to very different regions of the phase space. This phenomenon is closely linked to *mixing*, the concept that any set of points, no matter how small, will eventually spread uniformly in the phase space under the evolution. In fact, it is a simple exercise to show that (topological) mixing implies sensitive dependence on initial conditions.

In practice, this topological viewpoint is limited by the fact that the geometric structure of individual orbits can be extremely complicated. An important breakthrough in the theory of dynamical systems was the realisation (usually attributed to Kolmogorov) that complicated behaviour can be studied from a probabilistic point of view. In other words, instead of studying individual orbits, we want to study the long-term behaviour of typical orbits, where 'typical' is understood with respect to some dynamically relevant (invariant) measure. This is the setting of ergodic theory: the study of statistical properties of dynamical systems relative to a measure on the underlying space.

The measure-theoretic notions quantifying the degree of 'chaoticity' corresponding to the notions of sensitivity and topological mixing are *Lyapunov exponents* and *mixing rates*, respectively. The former measure the average exponential separation rate of nearby points, whereas the latter yield the speed with which the system loses its 'memory' of the initial state. In practice, the actual state of the system is often hidden, and can only be observed via certain functions of the state, the so-called *observables.* Viewed through given observables, the statistical correlation between initial and future states is described by *correlation functions.* The asymptotics of the decay of these correlation functions therefore describes the speed with which the evolution of the system decorrelates states. In general, this speed depends on the chosen observables, but by restricting the space of observables (by imposing a certain regularity) it is possible to obtain a specific (for example, exponential or polynomial) decay rate for 'typical' observables. This rate is also referred to as the mixing rate and provides a measure for the degree of chaoticity.

Having two distinct quantifiers of chaoticity, it is tempting to explore whether and how the two are related, and to what extent information on one has significance for the other. This question is of great practical interest, as in real world experiments correlation functions are often directly accessible from data, whereas Lyapunov exponents are notoriously hard to determine. It is a common perception in the physics literature that for chaotic low-dimensional systems exhibiting exponential decay of correlations, the decay rate and the Lyapunov exponents are related[1].

A powerful approach for the study of the decay of correlation functions consists of reformulating the problem in terms of spectral properties of the so-called *transfer operator*, also known as the (Ruelle-)Perron-Frobenius operator. This operator, originally developed in statistical mechanics [**44, 71**], describes how a distribution of initial points evolves under the action of the underlying dynamics, thus providing a global representation of a system's dynamics. Moreover, its spectrum yields insight into the ergodic-theoretic properties of the underlying system (see, for example, [**7, 17**] and references therein). In a certain setting, provided the system is exponentially mixing, the exponential mixing rate is determined by the size of the spectral gap of the associated transfer operator. In most cases, it is difficult to obtain the actual value for the exponential mixing rate, but explicit lower bounds have been obtained employing the transfer operator in various settings [**42, 51, 62, 76**].

While typical observables decay with the rate given by the mixing rate, faster exponential decay can occur if one chooses (nontypical) observables, that is, observables in certain subspaces of finite codimension. The spectrum of the transfer operator contains all possible exponential rates of correlation decay, and is often referred to as the correlation spectrum, see [**20**]. In a setting where the transfer operator is compact[2], its spectrum consists of isolated eigenvalues (which can accumulate at zero) together with zero itself, and these eigenvalues precisely correspond to the possible faster decay rates. Explicit upper bounds for the eigenvalue sequence (equivalently, lower bounds on the respective decay rates) were obtained in [**11, 12, 30**] for expanding systems in one and higher dimensions. Lower bounds on the eigenvalues are much harder to obtain, see however [**63, 64**].

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[1] In [**86**] it was even suggested to take correlation decay rates as meaningful estimates for Lyapunov exponents.

[2] For analytic expanding maps Ruelle [**72**] was the first to show compactness of the associated transfer operator when acting on certain spaces of holomorphic functions.

To illustrate the main questions presented in this work, let us first look at the well-known doubling map $x \mapsto 2x \bmod 1$ on the interval, the ergodic properties of which are fully understood. For analytic observables, the spectrum of the associated transfer operator consists of the eigenvalues $(1/2)^k$ for $k \in \mathbb{N}_0$ together with 0; thus the exponential mixing rate coincides with the Lyapunov exponent given by $\ln(2)$. Clearly, an equality of this type cannot hold for general exponentially mixing interval maps, as it is easy to construct simple maps with finite Lyapunov exponent but arbitrarily slow mixing rate (see, for example, [**38**]). However, it remains a valid question whether, conversely, there is a barrier for the speed of mixing in terms of the Lyapunov exponent for a meaningful class of exponentially mixing interval maps. More generally, one can ask whether there are interval maps (with a bounded number of smooth monotone branches) with arbitrarily large exponential mixing rate.

A wider range of questions emerges, if one observes that the 'same' map viewed on the circle, that is, the map $z \mapsto z^2$, is superexponentially mixing for analytic observables, as the spectrum of the associated transfer operator is the two-point set $\{0, 1\}$. One can then pose the question whether there are exponentially mixing circle maps with countably many nonzero eigenvalues in the spectrum of the corresponding transfer operator (as, for example, for the doubling map on the interval).

Motivated by the above, this thesis makes a contribution towards the understanding and explicit determination of possible exponential mixing rates and their relation with Lyapunov exponents, in the setting of expanding one-dimensional maps.

Chapter 1 is devoted to setting up a mathematical framework by introducing the relevant concepts and discussing the relations between mixing, correlation decay and the spectral properties of transfer operators, with a particular emphasis on a suitable choice of spaces of analytic observables. Some background material on spectral theory and a few technical proofs are deferred to the appendix.

In Chapter 2 we tackle the question of a possible relation between the two measures of chaoticity, Lyapunov exponents and mixing rates, in the setting of one-dimensional maps. More specifically, for piecewise linear expanding Markov maps observed via piecewise analytic functions, we show that the exponential mixing rate is bounded above by twice the Lyapunov exponent, that is, we establish lower bounds for the subleading eigenvalue of the corresponding transfer operator. In the proof we make use of the generalised transfer operator and the properties of topological pressure. We conclude the chapter by presenting numerical results suggesting that this bound cannot be expected to hold for nonlinear maps.

In view of the question whether this bound can be modified to hold in the nonlinear setting, we set out in Chapter 3 to construct a (nonlinear) expanding interval map the transfer operator of which has a prescribed eigenvalue and eigenfunction. It turns out that this construction yields a family of analytic expanding circle maps, for which the entire correlation spectrum can be determined explicitly using a suitable matrix representation of the associated transfer operator. In particular, for any $\lambda \in \mathbb{C}$ with

$|\lambda| < 1$ we obtain a circle map with correlation spectrum on the space of analytic observables consisting precisely of all nonnegative powers of λ and $\overline{\lambda}$ together with 0. These are the first examples of analytic circle maps with explicitly known nontrivial correlation spectra on the space of analytic functions. Moreover, viewed as interval maps, these maps provide counterexamples to a weak variant of a conjecture of Mayer on the reality of spectra.

The aim of Chapter 4 is to reveal the underlying structure of transfer operators associated to analytic expanding circle maps. For this, we first derive a natural representation of the respective (Banach space) adjoint operators, by viewing these as compressions of certain composition operators on spaces of holomorphic functions. For a special class of expanding circle maps arising from finite Blaschke products, this representation takes a particularly convenient form allowing us to deduce the entire spectrum of the corresponding (compact) transfer operators. Interestingly, these spectra are completely determined by the multipliers of the attracting fixed points of the Blaschke products. Moreover, the family of maps constructed in Chapter 3 belong to the class of finite Blaschke maps, and their spectra (previously computed from a matrix representation) are now simply explained.

The explicit knowledge of correlation spectra for Blaschke products now allows to answer questions about mixing rates and their relation to Lyapunov exponents. We present a family of full branch expanding interval maps (with a fixed number of branches) which exhibit arbitrarily fast exponential mixing but bounded Lyapunov exponents.

The thesis is concluded with a short summary and a selection of open questions following from our work.

By studying the correlation spectra for expanding one-dimensional maps we contribute to clarifying the relation between dynamical sensitivity and correlation decay. While our results indicate that a straightforward quantitative relation might not exist, the explicit determination of the spectra makes a range of interesting questions on possible mixing rates more accessible.

CHAPTER 1

Statistical properties of dynamical systems

This chapter sets the scene by providing background material and motivation for the research presented in this thesis. We start by briefly recalling concepts such as ergodicity, mixing and decay of correlations in the context of dynamical systems (Sections 1.1–1.2). We will then focus on systems enjoying exponential decay of correlations and establish the key link between the mixing rate and the spectrum of the transfer operator (Section 1.3). Section 1.4 discusses the transfer operator acting compactly on analytic function spaces and presents arguments frequently used in the later chapters. Material presented in this chapter is well known and serves as the basis for the chapters to follow.

1.1. Motivation: Chaos in topological dynamical systems

A *discrete-time dynamical system* consists of a nonempty set X and a transformation $T\colon X \to X$. The dynamics arises from the iteration of T on some initial point $x \in X$. The n-th iterate of T with $n \in \mathbb{N}$ is the n-fold composition $T^n = T \circ \cdots \circ T$ and the *(forward) orbit* of $x \in X$ is defined to be the infinite sequence $(T^n(x))_{n\in\mathbb{N}}$.

Depending on the purpose, the phase space X carries some additional structure and may, for example, be a topological space, a measure space, or a smooth manifold. The transformation T preserves the respective structure by being, for example, a continuous, measurable or smooth map. Throughout this thesis we shall only be interested in the dynamics of smooth and piecewise smooth one-dimensional maps with X a closed interval $I \subset \mathbb{R}$ or the complex unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

A primary goal of dynamical systems theory is to understand the long-term qualitative behaviour of (typical) orbits. Even in the seemingly easy setup of one-dimensional maps, orbits might be complicated, making the system appear 'chaotic'. While there is no general definition of chaos (see, for example, the discussion in [**67**, Ch. 3.5]), a general view is that a key ingredient for this phenomenon is sensitivity to initial conditions.

Before giving a precise definition, let us endow our dynamical system with a topological structure. By a *topological dynamical system* we mean a continuous map $T\colon X \to X$ on a topological space X, which for all our purposes will be a compact metric space (X, d).

Definition 1.1.1. A topological dynamical system on a metric space (X, d) is said to have *sensitive dependence on initial conditions* if there is an $\varepsilon > 0$ such that for every $x \in X$ and $\delta > 0$ there are $y \in X$ with $d(x, y) < \delta$ and $n \in \mathbb{N}$ such that $d(T^n(x), T^n(y)) > \varepsilon$.

Another phenomenon associated with chaotic motion is mixing, which, in the topological setting, is defined in the following way.

DEFINITION 1.1.2. A topological dynamical system is said to be *topologically mixing* if for any nonempty open sets U and V there exists an $n_0 \geq 1$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$.

It is straightforward to relate the above notions in the following way.

LEMMA 1.1.3. *Let $T\colon X \to X$ be a topological dynamical system on a compact metric space (X, d) consisting of more than one point. Then,*

topological mixing ⇒ sensitive dependence on initial conditions.

PROOF. Let T be topologically mixing and pick two distinct points $x_1, x_2 \in X$ and $\varepsilon < d(x_1, x_2)/4$. Then, for every open ball $U = B(x, \delta)$ around $x \in X$ with radius δ, there is an n such that $T^n(U) \cap V_1 \neq \emptyset$ and $T^n(U) \cap V_2 \neq \emptyset$, where $V_1 = B(x_1, \varepsilon)$ and $V_2 = B(x_2, \varepsilon)$. Hence, there are $y_1, y_2 \in U$ with $T^n(y_1) \in V_1$ and $T^n(y_2) \in V_2$. Clearly, these satisfy $d(T^n(y_1), T^n(y_2)) > 2\varepsilon$. It follows that $\max\{d(T^n(x), T^n(y_1)),\ d(T^n(x), T^n(y_2))\} > \varepsilon$. □

Going beyond the topological nature of the above lemma, it is tempting to ask for a more precise, quantitative relation between sensitive dependence and mixing. In particular, we want to explore in which way Lyapunov exponents (providing a quantitative measure for sensitive dependence) are related to mixing rates. Both notions require a measure-theoretical formulation, which will be introduced in the next section.

1.2. Lyapunov exponent, mixing and decay of correlations

Background on measure theory and probability theory can be found in [**36**]. We shall briefly recall the basic notions from ergodic theory. For a full introduction see, for example, the excellent book [**92**].

A probability space $(X, \mathcal{B}, \mu)$ is given by a set X, a probability measure μ and a σ-algebra $\mathcal{B}$ of μ-measurable subsets. To any such measure space one can associate the Banach spaces $L^p(X, \mu) = \{f\colon X \to \mathbb{C} : \int_X |f|^p \, d\mu < \infty\}$ with $p \geq 1$, and $L^\infty(X, \mu)$ the space of complex-valued essentially bounded measurable functions, with the usual respective norms $\|\cdot\|_p$. A transformation $T\colon X \to X$ is *measurable* if $B \in \mathcal{B}$ implies $T^{-1}(B) \in \mathcal{B}$. A measurable transformation $T\colon X \to X$ is called *nonsingular* if $\mu(T^{-1}(B)) = 0$ whenever $\mu(B) = 0$ for $B \in \mathcal{B}$, and it is called *measure-preserving* (or equivalently μ is *T-invariant*) if $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$. Finally, a measure-preserving transformation T (or the T-invariant measure μ) is called *ergodic* if $\mu(B) = 0$ or 1 for any set $B \in \mathcal{B}$ with $\mu(T^{-1}(B) \triangle B) = 0$. This definition implies that an ergodic system is irreducible, in the sense that it is impossible to subdivide the phase space into smaller parts on which the system can be studied separately.

The first major result in ergodic theory is the famous Birkhoff ergodic theorem, which asserts that asymptotic time averages exist for almost every point with respect to an invariant measure μ and coincide with the space average, provided μ is ergodic.

THEOREM 1.2.1 (Birkhoff, 1931). *Let $T\colon X \to X$ be measure-preserving on a probability space $(X, \mathcal{B}, \mu)$, and $f \in L^1(X, \mu)$. Then there is $f^* \in L^1(X, \mu)$ such that $(1/n)\sum_{k=0}^{n-1} f(T^k(x))$ converges to $f^*(x)$ for μ-a.e.*[1] *x as $n \to \infty$. Moreover, if μ is ergodic, then $f^*(x) = \int_X f\, d\mu$ for μ-a.e. x.*

Returning to the original goal of obtaining a quantitative version of Lemma 1.1.3, we introduce the Lyapunov exponent as a measure for sensitive dependence. If T is a differentiable map on X, where X is an interval or circle, then the *pointwise Lyapunov exponent* of a point $x \in X$, if it exists, measures the exponential expansion rate along the orbit of x and is defined as

$$\Lambda_x = \lim_{n\to\infty} \frac{1}{n} \ln |(T^n)'(x)|. \tag{1.1}$$

Let μ be a T-invariant ergodic measure and assume that $\ln |T'| \in L^1(X, \mu)$, then Birkhoff's ergodic theorem implies that this limit exists and is the same for μ-almost every x, as

$$\Lambda_x = \lim_{n\to\infty} \frac{1}{n} \ln |(T^n)'(x)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |T'(T^i(x))| = \int_X \ln |T'|\, d\mu. \tag{1.2}$$

We then simply write $\Lambda = \Lambda_x$ for the *Lyapunov exponent* with respect to μ.

However, a given system may have many ergodic invariant measures which lack physical significance, meaning they do not characterise a sufficiently large set of points, or more formally, sets with full μ-measure may have zero Lebesgue measure m, for example, if μ is supported on a periodic orbit. Hence, one is interested in *absolutely continuous invariant probability (acip)* measures with respect to Lebesgue measure, that is $\mu(B) = 0$ whenever $m(B) = 0$. By the Radon-Nikodým theorem, every acip measure μ is uniquely determined by a nonnegative function f such that $\mu(B) = \int_B f\, dm$ for any $B \in \mathcal{B}$.

There is a large body of literature on existence (and uniqueness) of acip measures for various dynamical systems (see [**54**, Ch. III] and [**17**, Ch. 5] for an overview). The pioneering work in this area focussed on subshifts of finite type [**15, 70**], piecewise expanding C^2 interval maps [**47, 77**], and topologically mixing Markov maps [**2, 16**].

In view of Lemma 1.1.3, a suitable statistical property similar to topological mixing is measure-theoretical mixing, or simply mixing.

DEFINITION 1.2.2. A measure-preserving transformation T (or the T-invariant measure μ) is *mixing*[2] if

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{B}. \tag{1.3}$$

[1] A property holds for μ-almost every (a.e.) $x \in X$ if it only fails to hold on a set of zero μ-measure.

[2] In the literature, this notion is often referred to as *strong mixing*.

Intuitively, mixing means that any small patch in the phase space is eventually uniformly distributed in the entire phase space, in other words $T^{-n}(B)$ is spreading uniformly with respect to μ.

Many systems are mixing, and indistinguishable at this level of description. To develop finer mixing properties, we need to generalise the concept of mixing and introduce correlation functions.

DEFINITION 1.2.3. Given a transformation $T\colon X \to X$ preserving the measure μ, the *correlation function* is defined as

$$C_{f,g}(n) = \int_X f \cdot (g \circ T^n)\, d\mu - \int_X f\, d\mu \int_X g\, d\mu \quad (n \in \mathbb{N}), \tag{1.4}$$

for some observables $f \in L^1(X,\mu)$ and $g \in L^\infty(X,\mu)$.

If f and g are the characteristic functions of some measurable sets A and B, then the convergence of $C_{f,g}(n)$ to 0 as $n \to \infty$ gives precisely the definition of mixing.[3] A natural question is then how fast the correlation function of a given mixing transformation converges to zero as $n \to \infty$. The answer critically depends on the properties of the map T and on the regularity of the observables. In [**25**], it was demonstrated that even for highly chaotic systems (in that case, hyperbolic automorphisms of the torus with positive entropy), the decay rate can be faster than exponential, exponential, or polynomial, depending on the choice of observables.

However, if one restricts the space of allowed observables to certain subspaces of $L^1(X,\mu)$, it is possible to obtain specific rates. There is a vast mathematical literature on the different possible rates of mixing (for example, exponential, stretched-exponential or polynomial) occurring for various one- and higher-dimensional systems with different degree of expansivity or hyperbolicity, see [**6**] for a nice summary.

As most results in the literature concern upper bounds for the correlation function, saying that it decays with a certain rate usually means it decays *not slower* than with this rate. For the purpose of this work, we will be solely concerned with systems exhibiting exponential decay of correlations. The system is *exponentially mixing* or enjoys *exponential decay of correlations* on a certain subspace $V \subset L^1(X,\mu)$ if there is a $\gamma \in (0,1)$ such that for any $f, g \in V$,

$$C_{f,g}(n) = O(\gamma^n) \qquad \text{as } n \to \infty. \tag{1.5}$$

Ideally, one is interested in the actual mixing rate, that is, in both upper and lower bounds (on the rate). There is a considerable body of literature on lower bounds for the decay rate (see, [**7, 18, 42, 51, 62, 76, 78**], to name but a few), but to the best of the author's knowledge hardly any nontrivial upper bounds exist (see however [**63**], where an exponential upper bound for the decay rate of correlation functions of suspension semiflows is given). For obtaining a quantitative version of Lemma

[3]Further, one can show that μ is mixing if and only if $C_{f,g}(n) \to 0$ as $n \to \infty$ for any $f \in L^1(X,\mu)$ and $g \in L^\infty(X,\mu)$.

1.1.3, we will indeed require upper bounds on the rates, as we want to say that, for any 'typical' observables f and g in a certain subspace of $L^1(X,\mu)$, the correlation function decays *not faster* than with a certain exponential rate, and relate this rate to the Lyapunov exponent.

There are several tools available in the literature for showing exponential decay of correlations, see [**6, 7**] for an overview. One of the most common approaches, and the one we will pursue here, is a functional-analytic one, which relies on associating to T the so-called transfer operator and investigating its spectral properties. Other noteworthy approaches are based on Birkhoff cones [**51, 91**] or probabilistic coupling methods [**93**].

1.3. Rates of mixing and transfer operators

Let X be an interval or the circle and $T\colon X \to X$ a nonsingular map with respect to (normalised) Lebesgue measure m. One can associate to T a certain linear operator $\mathcal{L}$, known as the transfer operator, which can be defined in the following way.

DEFINITION 1.3.1. The *transfer operator* $\mathcal{L}\colon L^1(X,m) \to L^1(X,m)$ is defined by

$$\int_X \mathcal{L}f \cdot g\, dm = \int_X f \cdot (g \circ T)\, dm \tag{1.6}$$

for all $f \in L^1(X,m)$ and all $g \in L^\infty(X,m)$.

Equation (1.6) realises the idea that the initial density f is transformed to the density $\mathcal{L}f$ under the action of the map T. For this, note that taking $g = \chi_B$ to be the characteristic function of a set $B \in \mathcal{B}$, the mass of points landing in B under application of T is $\int_{T^{-1}(B)} f\, dm = \int_X f \cdot (\chi_B \circ T)\, dm = \int_B \mathcal{L}f\, dm$. The space $L^1(X,m)$ is a natural choice for the domain of $\mathcal{L}$, as its (normalised) nonnegative elements are the probability densities.

REMARK 1.3.2. Many authors refer to $\mathcal{L}$ as the *Perron-Frobenius* operator and use the term 'transfer operator' for a more general version. For simplicity, throughout this thesis we will always call these operators transfer operators.

Note that $\mathcal{L}$ is a positive bounded operator[4] with $\|\mathcal{L}\| = 1$. Its spectrum $\sigma(\mathcal{L})$ has a dynamical interpretation in terms of the ergodic-theoretical properties of T. The densities of acip measures μ are exactly the probability densities that are fixed by $\mathcal{L}$, that is, eigenfunctions of $\mathcal{L}$ with eigenvalue 1. If such an acip measure exists and is ergodic, then it must be unique [**46**, Thm. 4.2.2], meaning the geometric multiplicity of the eigenvalue 1 is equal to 1. Moreover, its algebraic multiplicity is also[5] 1. Henceforth, we will call an eigenvalue *simple* if its algebraic multiplicity is 1. If, in addition,

[4]One can check that $\|\mathcal{L}f\|_1 \le \|f\|_1$ for any $f \in L^1(X,m)$. By positivity, meaning that $f \ge 0$ implies $\mathcal{L}f \ge 0$, it follows that $\|\mathcal{L}f\|_1 = \|f\|_1$ for any $f \ge 0$.

[5]Suppose, for a contradiction, that the algebraic multiplicity is $n = 2$, then $(I - \mathcal{L})^2 g = 0$ and $(I - \mathcal{L})g \ne 0$ for some g. It follows that $g - \mathcal{L}g = c\varrho$ for some $c \ne 0$, where ϱ is the density of the unique acip measure. Integrating this equation and using $\int_X \mathcal{L}f\, dm = \int_X f\, dm$, leads to the contradiction $0 = c\int \varrho\, dm$. A similar argument applies for $n > 2$.

the acip measure has density function $f > 0$ and is mixing, then $\mathcal{L}$ has exactly one eigenvalue on the unit circle, equal to 1. See [**17**, Ch. 3] and [**46**, Ch. 4] for easily accessible accounts of these and related facts.

REMARK 1.3.3. As an aside, equation (1.6) defines the Banach space adjoint of the transfer operator, known as the *Koopman operator*, defined on $L^\infty(X,m)$ and given by $g \mapsto g \circ T$. It can be used for formulating such concepts as ergodicity and mixing, and characterises these in terms of its spectral properties, see [**46**, §4.4] or [**92**, §1.7].

By imposing analyticity on the map T, we will show in Chapter 4 that it is possible to define such operators, in this context known as a *composition operators*, on certain analytic function spaces. In a particular setting where the associated transfer operator is compact, its entire spectrum will then be deduced from the spectra of these composition operators.

Now $\mathcal{L}$ can be directly linked to the correlation function defined in (1.4). Throughout this chapter, we will freely make use of facts and notions from basic spectral theory, which are summarised in Appendix A.

For convenience, we make the following standing assumption for the rest of this chapter.

(AS1) Let $T\colon X \to X$ be nonsingular with respect to m and assume that it possesses a unique acip measure μ with density $\varrho \in L^1(X,m)$ bounded away from zero and infinity, and normalised so that $\int_X \varrho \, dm = 1$.

Note that this implies $L^p(X,m) = L^p(X,\mu)$ for all $1 \leq p \leq \infty$. Using the continuous projection $P\colon L^1(X,m) \to L^1(X,m)$ given by

$$Pf = \left(\int_X f dm\right) \varrho \tag{1.7}$$

and $M_\varrho\colon L^1(X,m) \to L^1(X,m)$ the operator of multiplication with ϱ,

$$M_\varrho f = \varrho \cdot f, \tag{1.8}$$

we can rewrite[6] the correlation function (1.4) as

$$C_{f,g}(n) = \int_X g \cdot (\mathcal{L}^n - P)\,(M_\varrho f)\, dm \tag{1.9}$$

for all $f \in L^1(X,m)$ and $g \in L^\infty(X,m)$.

---

[6]By (1.6) we have on the one hand

$$\int_X f \cdot (g \circ T^n)\, d\mu = \int_X (g \circ T^n) \cdot f \varrho \, dm. = \int_X g \cdot \mathcal{L}^n(f\varrho)\, dm,$$

and on the other hand

$$\int_X f \, d\mu \int_X g \, d\mu = \int_X g \left(\int_X f \varrho \, dm\right) \varrho \, dm = \int_X g \cdot P(f\varrho)\, dm.$$

1.3.1. Spectral gap implies exponential mixing. The proofs for all results stated in this section and in Section 1.3.2 are somewhat technical, and are therefore deferred to Appendix B.

In order to define an adequate notion of rate of correlation decay for the system (also referred to as the mixing rate), we restrict $\mathcal{L}$ to certain invariant subspaces V of $L^1(X,m)$ on which $\mathcal{L}$ is quasicompact[7]. Recall that a quasicompact operator has essential spectral radius strictly less than its spectral radius. As we shall see, this property will guarantee that the correlation function in (1.9) decays exponentially for observables in V. To this end, we make the following standing assumption.

(AS2) Let T satisfy (AS1) and let $\mathcal{L}$ be its associated transfer operator. Assume that V is an $\mathcal{L}$-invariant subspace, densely and continuously embedded in $L^1(X,m)$ such that $\mathcal{L}$ restricted to V is quasicompact.

The following lemma summarises some well-known spectral properties of $\mathcal{L}\colon V\to V$ satisfying (AS2).

LEMMA 1.3.4. *Let T and V satisfy* (AS2). *Then the unique invariant acip density ϱ is in V. The spectral radius of $\mathcal{L}\colon V\to V$ is* 1, *and if additionally the acip measure is mixing, then the only spectral point on the unit circle is* 1, *which is a simple eigenvalue.*

Let us now define the exponential mixing rate on V. For simplicity and as it encompasses all the settings considered in this thesis, we assume V to be a subspace of $L^\infty(X,m)$.

DEFINITION 1.3.5. Let T and V satisfy (AS2) with V a subspace of $L^\infty(X,m)$. The *exponential rate of mixing on* V is defined as

$$\alpha_V = -\ln\sup\left\{\limsup_{n\to\infty}|C_{f,g}(n)|^{1/n} : f,g\in V\right\}. \tag{1.10}$$

We are now in a position to establish the key connection between the exponential mixing rate on V and the spectrum of the operator $\mathcal{L}$. The assumption that $\mathcal{L}$ restricted to V is quasicompact guarantees exponential mixing, that is $\alpha_V>0$.

PROPOSITION 1.3.6. *Let T and V satisfy* (AS2) *with V a subspace of $L^\infty(X,m)$, and $M_\varrho(V)\subseteq V$. Additionally, assume that the unique acip measure is mixing. Suppose that $\mathcal{L}\colon V\to V$ is quasicompact. Then*

$$\alpha_V \geq -\ln\sup\{|\lambda| : \lambda\in\sigma(\mathcal{L})\setminus\{1\}\} > 0.$$

In certain cases[8], in particular in those relevant for this work (when $\mathcal{L}$ is compact or V is the space of bounded variation), the mixing rate α_V is exactly determined by the size of the spectral gap of $\mathcal{L}\colon V\to V$, that is

$$\alpha_V = -\ln\sup\{|z| : z\in\sigma(\mathcal{L})\setminus\{1\}\}. \tag{1.11}$$

[7]Note that usually $\mathcal{L}$ is not quasicompact on $L^1(X,m)$.

[8]For V satisfying (AS2), and assuming that $M_\varrho(V)=V$ and that there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of eigenvalues of $\mathcal{L}\colon V\to V$ such that $\lim_{n\to\infty}|\lambda_n| = \sup\{|z| : z\in\sigma(\mathcal{L})\setminus\{1\}\}$ (see, for example, [**7**, Thm. 1.5 and 2.5] or [**22**]), one can show that $\alpha_V \leq -\ln\sup\{|z| : z\in\sigma(\mathcal{L})\setminus\{1\}\}$.

This implies that

$$\alpha_V = -\ln \max\{|\lambda_2(\mathcal{L})|, \rho_{ess}(\mathcal{L})\},$$

where $\lambda_2(\mathcal{L})$ is the second largest eigenvalue of $\mathcal{L}$ in modulus, and $\rho_{ess}(\mathcal{L})$ is the essential spectral radius of $\mathcal{L}$. The proof of (1.11) depends on the space V, and will be provided in the next section in the case of compact $\mathcal{L}\colon V \to V$ (which implies $\rho_{ess}(\mathcal{L}) = 0$).

1.3.2. Beyond the spectral gap. The size of the spectral gap, governed either by the second largest eigenvalue in modulus or by the essential spectral radius, determines the exponential mixing rate for typical observables from certain subspaces V of $L^1(X, m)$. However, assuming there are further (isolated) eigenvalues, a faster exponential rate of correlation decay determined by $|\lambda_n(\mathcal{L})|$ can occur, if one chooses (nontypical) observables in certain subspaces of finite codimension. These observables do not 'feel' the rates corresponding to the first $n-1$ eigenvalues, that is, they are in a subspace with vanishing spectral projections corresponding to the eigendirections of $\{\lambda_1(\mathcal{L}), \ldots, \lambda_{n-1}(\mathcal{L})\}$. In case the transfer operator $\mathcal{L}\colon V \to V$ is compact, the essential spectrum reduces to the origin, and the spectrum consists solely of eigenvalues, together with zero. The following lemma, valid for any compact operator L, and its corollary for transfer operators allow to make the above statement on faster decay rates precise.

LEMMA 1.3.7. *Let $L\colon V \to V$ be a compact operator on a Banach space V, with eigenvalue sequence $(\lambda_n(L))_{n\in\mathbb{N}}$, ordered by decreasing modulus, with repetitions according to algebraic multiplicity. Let V^* be the topological dual of V. Then, for $n \in \mathbb{N}$,*

$$|\lambda_n(L)| = \inf_{\substack{W_n \subseteq V \\ \operatorname{codim} W_n < n}} \sup \left\{ \limsup_{k\to\infty} |g(L^k f)|^{1/k} : f \in W_n, g \in V^* \right\}. \tag{1.12}$$

Simply put, for a fixed n the supremum on the right-hand side of (1.12) yields the slowest contraction rate, that is, largest eigenvalue, on a subspace W_n, and the infimum maximises this rate by selecting the W_n with the $n-1$ eigendirections corresponding to the largest eigenvalues removed.

Now, under some conditions, this lemma allows to relate all exponential decay rates of the correlation functions $C_{f,g}(n)$ for observables in $V \times V$ to the (nonzero) eigenvalues of a compact transfer operator $\mathcal{L}\colon V \to V$.

COROLLARY 1.3.8. *Let $T\colon X \to X$ and V be as in Proposition 1.3.6. Further, assume that $M_\varrho(V) = V$ and $\mathcal{L}\colon V \to V$ is compact with eigenvalue sequence $(\lambda_n(\mathcal{L}))_{n\in\mathbb{N}}$. Then, for $n > 1$,*

$$|\lambda_n(\mathcal{L})| = \inf_{\substack{W_n \subseteq V \\ \operatorname{codim} W_n < n-1}} \sup \left\{ \limsup_{k\to\infty} |C_{f,g}(k)|^{1/k} : f \in W_n, g \in V \right\}. \tag{1.13}$$

Thus, the mixing rate on V is

$$\alpha_V = -\ln |\lambda_2(\mathcal{L})|. \tag{1.14}$$

REMARK 1.3.9. The condition $M_\varrho(V) = V$ will be true for all subspaces under consideration in this work. In fact, this is implied by the assumption (AS1), as the unique invariant density ϱ is bounded away from zero and infinity. In the next chapters, we will mainly focus on spaces of holomorphic functions $V = H^\infty(U)$ or $H^2(U)$ (considered in Chapters 3 and 4) on a neighbourhood $U \subset \mathbb{C}$ containing X, which can be chosen such that the extension of ϱ to U is still bounded away from zero and infinity, implying $M_\varrho(V) = V$.

As an aside, while the assumption (AS1) is generally satisfied for (topologically mixing) uniformly expanding systems, it excludes several well-studied non-uniformly expanding systems, such as smooth interval or circle maps having a neutral fixed point or critical points, see [**53**] and the references therein for an overview. Such maps can possess a unique acip measure, albeit with density not bounded away from ∞. These maps, however, do not fall into the scope of this work, in which we are mainly concerned with uniformly expanding maps, giving rise to compact transfer operators on holomorphic function spaces and exponential decay of correlations.

In view of Corollary 1.3.8, knowledge of the spectrum of $\mathcal{L}\colon V \to V$, also known as the *correlation spectrum* (see [**20**]), is useful, as it determines all possible exponential correlation decay rates.

Moreover, it is well known (see, for example, [**73, 74**]) that the correlation spectrum is directly connected to the physically relevant *power spectrum* $\hat{C}_{f,g}$, that is the Fourier transform of the correlation function,

$$\hat{C}_{f,g}(\omega) = \sum_{n=0}^{\infty} e^{-in\omega} C_{f,g}(n) \quad (\omega \in \mathbb{C}).$$

The analytic structure of $\hat{C}_{f,g}$ is related to the analytic structure of the resolvent of the transfer operator as, with $z = e^{i\omega}$, we have

$$\begin{aligned}
\hat{C}_{f,g}(\omega) &= \sum_{n=0}^{\infty} z^{-n} C_{f,g}(n) \\
&= \int_X g \cdot \sum_{n=0}^{\infty} z^{-n} (\mathcal{L}^n - P)(M_\varrho f)\, dm \\
&= \int_X g \cdot z\, (zI - (\mathcal{L} - P))^{-1} (M_\varrho f)\, dm
\end{aligned}$$

for $|z|^{-1} \|\mathcal{L} - P\|_{V\to V} < 1$, where we have used $\mathcal{L}^n - P = (\mathcal{L} - P)^n$ for $n > 0$ (as P satisfies $\mathcal{L}P = P\mathcal{L} = P$) and the Neumann series expansion of the resolvent $(zI - (\mathcal{L} - P))^{-1} = \sum_{n=0}^{\infty} z^{-n-1}(\mathcal{L} - P)^n$, see, for example, [**87**, Thm. 3.1, Ch. V]. If $\mathcal{L}\colon V \to V$ is compact, then by [**87**, Cor. 10.3, Ch. V] the resolvent has a meromorphic extension to $\mathbb{C}\setminus\{0\}$, and hence the power spectrum $\hat{C}_{f,g}$ has a meromorphic extension to $\mathbb{C}$. The poles of $\hat{C}_{f,g}$ are called *resonances* and are given by i times the logarithm of the nonzero eigenvalues of $\mathcal{L} - P$. To summarise, the eigenvalues determine the resonances, and hence provide insights into the short-term behaviour of the system via the Fourier modes of the correlation function.

1.4. Transfer operators on analytic function spaces

Since in subsequent chapters $T\colon X \to X$ will be an analytic (or piecewise analytic) expanding map on an interval or circle, it will be advantageous to consider the associated transfer operator $\mathcal{L}$ on suitable spaces of analytic (holomorphic) functions on a neighbourhood U of X with some prescribed boundary behaviour. Historically, Ruelle [**72**] was the first to show, among other things, that for certain analytic expanding maps on compact $X \subset \mathbb{C}^d$, the associated transfer operator preserves and acts compactly on Banach spaces consisting of functions holomorphic on a certain neighbourhood U of X, which extend continuously to the boundary of U. Other authors have considered transfer operators on different spaces of holomorphic functions, examples of which are Hardy spaces, see, for example, [**58**], or Bergman spaces [**11, 37**]. Under mild assumptions it is possible to show that the spectrum of $\mathcal{L}$ does not depend on the particular choice of holomorphic function space [**12**].

For all our purposes we will consider Hardy spaces, and for simplicity of the next argument we will restrict ourselves in this section to the following one.

Definition 1.4.1. For U an open subset of $\mathbb{C}$, we write

$$H^\infty(U) = \left\{ f\colon U \to \mathbb{C} \,:\, f \text{ holomorphic and } \sup_{z\in U} |f(z)| < \infty \right\}$$

for the Banach space of bounded holomorphic functions on U equipped with the norm $\|f\|_{H^\infty(U)} = \sup_{z\in U} |f(z)|$.

For an analytic expanding full branch interval map $T\colon I \to I$, we will show that the associated transfer operator is well defined and compact on certain spaces $H^\infty(U)$. The key ingredient of the proof of this statement is a factorisation argument. In the following chapters this argument will be adapted to the settings of expanding Markov maps (Chapter 2) and analytic expanding circle maps (Chapters 3 and 4).

We first define a *partition* of a closed interval I to be a finite collection of closed intervals $\{I_1, \dots, I_K\}$ with disjoint interiors, that is, $\mathrm{int}(I_k) \cap \mathrm{int}(I_l) = \emptyset$ for $k \neq l$, such that $\bigcup_{k=1}^K I_k = I$.

Definition 1.4.2. Let $\{I_1, \dots, I_K\}$ be a partition of I. A transformation $T\colon I \to I$ is called an *analytic full branch map* if for all k

(i) $T_k = T|_{\mathrm{int}(I_k)}$ is a (real) analytic diffeomorphism,
(ii) $\mathrm{cl}(T(I_k)) = I$, and
(iii) the inverse Φ_k of T_k can be analytically extended to $\Phi_k\colon I \to I_k$.

Each T_k is called a *branch* of T, and the corresponding inverse Φ_k is referred to as an *inverse branch*. The map T is called *expanding* if $|T'(x)| > 1$ for all $x \in \bigcup_{k=1}^K \mathrm{int}(I_k)$.

It is not difficult to see from (1.6) using a change of variables, that for any analytic full branch map $T\colon I \to I$ the transfer operator $\mathcal{L}\colon L^1(I,m) \to L^1(I,m)$ can be written as

$$\mathcal{L}f = \sum_{k=1}^{K} W_k \cdot (f \circ \Phi_k), \tag{1.15}$$

with $W_k = \Phi_k'$ if $\Phi_k' > 0$ and $W_k = -\Phi_k'$ otherwise. More general transfer operators are given by (1.15) with other suitable weight functions W_k, see Chapter 2.

With slight abuse of notation we keep writing T and Φ_k for the respective analytic extensions to some bounded domain[9] $U \subset \mathbb{C}$ containing I.

Notation 1.4.3. As usual, we write $\mathrm{cl}(U)$ to denote the closure of U in $\mathbb{C}$. Given two open subsets U and V of $\mathbb{C}$ we write

$$U \Subset V$$

if $\mathrm{cl}(U)$ is a compact subset of V.

Assuming that T is expanding, all inverse branches Φ_k are contractions on I. We can thus choose domains U and U' containing I such that

$$\Phi_k(U) \subset U' \Subset U \text{ for all inverse branches } \Phi_k, \tag{1.16}$$

see, for example, [**12**, Lem. 2.4]. For suitable domains U containing I and $W_k \in H^\infty(U)$, we will show that the operator $\mathcal{L}$ in (1.15) leaves the subspace $H^\infty(U)$ of $L^1(I,m)$ invariant. Moreover, $\mathcal{L}$ acts compactly on these spaces. Both facts can be seen from the following factorisation of the operator.

1.4.1. Factorisation argument. Observe that in (1.15) the argument of $f \in H^\infty(U)$, that is $\Phi(z)$, is contained in the smaller domain U' because of (1.16). We can thus use (1.15) to view $\mathcal{L}$ as an operator from the larger function space $H^\infty(U')$ to $H^\infty(U)$. Note that the space is 'larger' as analyticity is only required on a domain $U' \subset U$. We shall write $\tilde{\mathcal{L}}$ in order to distinguish this lifted operator from $\mathcal{L}$ on $H^\infty(U)$. The boundedness of $\tilde{\mathcal{L}}$ is the content of the next lemma.

Lemma 1.4.4. *Let $W_k \in H^\infty(U)$ for $k = 1, \dots, K$. Suppose that U and U' are domains in $\mathbb{C}$ such that $\Phi_k(U) \subset U' \subset U$ for all inverse branches Φ_k. Then $\tilde{\mathcal{L}}$ given by* (1.15) *maps $H^\infty(U')$ continuously to $H^\infty(U)$.*

Proof. Set $W = \sup_{z\in U} \sum_{k=1}^{K} |W_k(z)| < \infty$ and note that

$$|(\tilde{\mathcal{L}}f)(z)| \leq \sum_{k=1}^{K} |W_k(z)||f(\Phi_k(z))| \leq W \, \|f\|_{H^\infty(U')} \quad (z \in U).$$

Hence, $\|\tilde{\mathcal{L}}f\|_{H^\infty(U)} \leq W \, \|f\|_{H^\infty(U')}$, so $\tilde{\mathcal{L}}$ is continuous. □

Choosing $U' = U$ in the previous lemma shows that $\mathcal{L}\colon H^\infty(U) \to H^\infty(U)$ is a well-defined continuous operator.

[9]By a domain we mean a nonempty connected open subset of $\mathbb{C}$.

We next assume that $U' \Subset U$ as in (1.16) and introduce the bounded embedding operator J which maps the smaller space $H^\infty(U)$ injectively into the larger space $H^\infty(U')$. To be precise, $J\colon H^\infty(U) \to H^\infty(U')$ is given by

$$(Jf)(z) = f(z) \quad \text{for } z \in U'. \tag{1.17}$$

The two operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are related by restriction, that is,

$$\mathcal{L} = \tilde{\mathcal{L}}J\,. \tag{1.18}$$

This relation is represented by the following diagram:

$$\begin{tikzcd} H^\infty(U') \arrow[dr, "\tilde{\mathcal{L}}"] & \\ H^\infty(U) \arrow[u, hook, "J"] \arrow[r, "\mathcal{L}"'] & H^\infty(U) \end{tikzcd} \tag{1.19}$$

Note that the factorisation above disentangles the intricacies of the map contained in $\tilde{\mathcal{L}}$ from its general expansiveness contained in J. Moreover, the embedding J is compact by an application of Montel's theorem (see, for example, [**23**, Thm. 2.9, Ch. 7]). Hence, as $\tilde{\mathcal{L}}$ is continuous, the factorisation (1.18) yields the following result.

Proposition 1.4.5. *The operator $\mathcal{L}\colon H^\infty(U) \to H^\infty(U)$ from* (1.15) *is compact.*

1.4.2. Approximation argument. In some cases (for example, piecewise linear Markov maps discussed in Chapter 2 or the family of expanding analytic circle maps considered in Chapter 3), one can show that the associated transfer operator $\mathcal{L}$ leaves certain finite-dimensional spaces invariant, and restricted to these spaces possesses a triangular matrix representation. More formally, $\mathcal{L}$ can be approximated by finite rank operators, whose matrix representations admit triangular form. The next lemma connects the spectrum of $\mathcal{L}$ with that of its finite rank approximations. As this lemma will be used several times, we shall state it for general Banach spaces.

Lemma 1.4.6. *For Banach spaces V and V', let $\mathcal{L}\colon V \to V$ be a bounded linear operator which admits a factorisation of the form* (1.18) *with $\tilde{\mathcal{L}}\colon V' \to V$ and $J\colon V \to V'$ bounded. Let $P_N\colon V \to V$ denote a continuous projection onto an N-dimensional subspace $H_N = P_N(V)$. If*

(a) $\mathcal{L}(H_N) \subseteq H_N$ *for all* $N \in \mathbb{N}_0$*, and*
(b) $\lim_{N\to\infty} \|J - JP_N\|_{V\to V'} = 0$,

then $\mathcal{L}$ is compact and its spectrum is given by

$$\sigma(\mathcal{L}) = \mathrm{cl}\Big(\bigcup_{N\in\mathbb{N}_0} \sigma(\mathcal{L}|_{H_N})\Big).$$

Proof. Clearly $\sigma(\mathcal{L}|_{H_N}) \subseteq \sigma(\mathcal{L})$. Since $\mathcal{L}(H_N) \subseteq H_N$ we have $\mathcal{L}P_N = P_N\mathcal{L}P_N$ for every $N \in \mathbb{N}_0$. Using the factorisation (1.18) we see that

$$\begin{aligned}\|\mathcal{L} - P_N\mathcal{L}P_N\|_{V\to V} &= \|\mathcal{L} - \mathcal{L}P_N\|_{V\to V} = \|\tilde{\mathcal{L}}J - \tilde{\mathcal{L}}JP_N\|_{V\to V} \\ &= \|\tilde{\mathcal{L}}(J - JP_N)\|_{V\to V} \\ &\leq \|\tilde{\mathcal{L}}\|_{V'\to V}\|J - JP_N\|_{V\to V'}\,,\end{aligned}$$

which, using boundedness of $\tilde{\mathcal{L}}$ and *(b)*, implies

$$\lim_{N\to\infty} \|\mathcal{L} - P_N\mathcal{L}P_N\|_{V\to V} = 0\,. \tag{1.20}$$

Since $P_N\mathcal{L}P_N$ is a finite rank operator for every N, the limit above implies that $\mathcal{L}$ is compact. Clearly, the nonzero eigenvalues of each $P_N\mathcal{L}P_N$ are exactly the nonzero eigenvalues of $\mathcal{L}|_{H_N}$. By (1.20) and an abstract spectral approximation result (see [**27**, XI.9.5]), every nonzero eigenvalue $\lambda \in \sigma(\mathcal{L})$ is a limit of some sequence (λ_N) with $\lambda_N \in \sigma(\mathcal{L}|_{H_N})$, which proves the remaining equality. □

For later use, we shall verify the approximation assumption (b) of the previous lemma for the domains $U = D_R$ and $U' = D_r$ satisfying (1.16) for $\mathcal{L}\colon H^\infty(U) \to H^\infty(U)$ given in (1.15), where D_r and D_R denote two concentric open disks in $\mathbb{C}$ centred at a point z_0 with respective radii $0 < r < R$. We introduce a projection operator defined as follows: given an analytic function f in $H^\infty(D_R)$ and an integer N, we define P_Nf to be the function given by the truncated Taylor series expansion

$$(P_Nf)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n\,. \tag{1.21}$$

Clearly, P_N is a projection operator on $H^\infty(D_R)$. It turns out that the finite rank operators JP_N approximate the embedding $J\colon H^\infty(D_R) \to H^\infty(D_r)$ in (1.17) for large N in a strong sense, which is the statement of the following lemma.

Lemma 1.4.7. *For the setting above, the assumption (b) from Lemma 1.4.6 holds:*

$$\lim_{N\to\infty} \|J - JP_N\|_{H^\infty(D_R)\to H^\infty(D_r)} = 0\,.$$

Proof. By Cauchy's integral theorem, we have for any $f \in H^\infty(D_R)$ and $z \in D_r$

$$\begin{aligned}f(z) - (P_Nf)(z) =& \frac{1}{2\pi i}\int_\gamma \sum_{n=N+1}^{\infty} \frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta \\ =& \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\frac{(z - z_0)^{N+1}}{(\zeta - z_0)^{N+1}}\, d\zeta\,,\end{aligned}$$

where the contour γ is the positively oriented boundary of a disk centred at z_0 with radius R' lying strictly between r and R. It follows that the norm of $J - JP_N$ viewed as an operator from $H^\infty(D_R)$ to $H^\infty(D_r)$ satisfies

$$\|J - JP_N\|_{H^\infty(D_R)\to H^\infty(D_r)} \leq \frac{R'}{R' - r}\left(\frac{r}{R'}\right)^{N+1}\,,$$

from which the statement follows. □

REMARK 1.4.8. The above lemma implies that J is compact, and hence using the factorisation (1.16) shows that $\mathcal{L}$ in (1.15) is compact, which is an alternative way of proving compactness of $\mathcal{L}$ without invoking Montel's theorem.

By establishing compactness of the transfer operator in an analytic setup we have set the stage for the investigations in the next chapters. There, the different settings will require modified proofs of compactness, but all these modifications will follow the structure presented above and build on these results. The presented link between the spectrum of the transfer operator and the mixing rates will enable us to prove certain bounds on these rates in terms of other dynamical quantities such as Lyapunov exponents. Further, for a particular class of dynamical systems we will be able to determine the entire spectrum and hence obtain all possible exponential mixing rates.

CHAPTER 2

Relation between mixing rates and Lyapunov exponents

Following the discussion in the introduction and in Chapter 1, the purpose of this chapter is to approach the problem of obtaining a quantitative version of Lemma 1.1.3, or more specifically, to relate the exponential mixing rate and the Lyapunov exponent of simple expanding Markov maps.

It is a common perception in the physics literature that in chaotic low-dimensional systems enjoying exponential decay of correlations there should be an intuitive relationship between Lyapunov exponents and correlation decay rates. In [**5**] an exact relation between correlation decay and (generalised) Lyapunov exponents was conjectured, and in [**86**] it was even suggested to take correlation decay rates as a meaningful approximation for Lyapunov exponents.

On the other hand, one may argue that both quantities probe entirely different and independent aspects of a dynamical system. The mixing rate is determined by the size of the spectral gap of the associated transfer operator (given by the subleading eigenvalue or the essential spectral radius). In contrast, considering a certain one-parameter family of operators containing the transfer operator, the Lyapunov exponent is governed by the sensitivity of the leading eigenvalue of these operators with respect to the parameter (see Lemma 2.2.8 for precise statements).

In this chapter, we shall explore a possible relation in the simple setup of piecewise linear expanding Markov maps. After motivating this relation in Section 2.1, we will show in Section 2.2 that for these maps observed via piecewise analytic functions, the decay rate is bounded above by twice the Lyapunov exponent; that is, we establish lower bounds for the subleading eigenvalue of the corresponding transfer operator. In Section 2.3 we shall provide numerical evidence that such bounds break down once we consider general nonlinear expanding Markov maps, but certain relations still hold if the transfer operator is considered on the space of bounded variation.

The results of this chapter are contained in the publication [**85**].

2.1. A pedestrian approach

The problem we want to address can be illustrated by a basic textbook example, considered, for instance, in [**5**]. Take a linear full branch map (see Figure 2.1) on the unit interval $I = [0, 1]$, that is, a map $T\colon I \to I$ with constant slope γ_k on each I_k, where $\{I_1, \dots, I_K\}$ is a finite partition of I into $K \geq 2$ closed intervals. This is a simple instance of Definition 1.4.2.

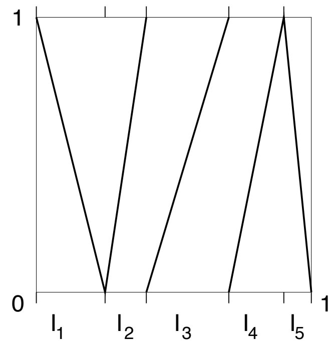

FIGURE 2.1. Diagrammatic view of a linear full branch map.

The unique acip measure is given by the Lebesgue measure and the Lyapunov exponent (1.2) with respect to this measure can be expressed in terms of the slopes as

$$\Lambda = \sum_{k=1}^{K} |I_k| \ln |\gamma_k|, \tag{2.1}$$

with $|I_k| = 1/|\gamma_k|$ denoting the size of the interval I_k.

The exponential mixing rate is determined by the second largest eigenvalue in modulus of the associated transfer operator restricted to a space of sufficiently smooth functions, see Corollary 1.3.8. In this setting it is well known (see, for example, [**61**]) that eigenfunctions of the transfer operator are given by polynomials and that the corresponding eigenvalues ν_m can be expressed as

$$\nu_m = \sum_{k=1}^{K} \frac{1}{|\gamma_k|} \frac{1}{\gamma_k^m} = \sum_{k=1}^{K} |I_k| \frac{1}{\gamma_k^m} \qquad (m \geq 0), \tag{2.2}$$

with largest eigenvalue $\lambda_1 = \nu_0 = 1$. If all slopes γ_k have the same sign then (2.2) defines a monotonic sequence $(|\nu_m|)_{m\geq 0}$ and the subleading eigenvalue λ_2, being the second largest in modulus, is equal to ν_1. If the slopes have different sign then the subleading eigenvalue is given either by ν_1 or by ν_2, depending on which of $|\nu_1|$ or ν_2 is larger, that is, $|\lambda_2| = \max\{|\nu_1|, \nu_2\}$. Correlation functions of sufficiently smooth observables decay typically at an exponential rate $\alpha = -\ln |\lambda_2|$. Since $|\lambda_2| \geq \nu_2 > 0$ we obtain an upper bound $\alpha \leq -\ln \nu_2$ for the decay rate, which can now be related to the Lyapunov exponent (2.1). If we apply Jensen's inequality[1] to the convex function $\varphi(x) = -\ln(x)$ we end up with

$$\alpha \leq -\ln \left(\sum_{k=1}^{K} |I_k| \frac{1}{\gamma_k^2} \right) \leq \sum_{k=1}^{K} |I_k| \left(-\ln \frac{1}{\gamma_k^2} \right) = 2\Lambda \,. \tag{2.3}$$

The estimate of the decay rate in (2.3) is based on ν_2, which contains positive terms only, even if the slopes have different signs. As a result the upper bound is given by twice the Lyapunov exponent. If all slopes have the same sign, say $\gamma_k > 1$, then

[1]Jensen's inequality can be stated as follows. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex, $x_1, \ldots, x_n \in \mathbb{R}$, and $a_1, \ldots, a_n$ positive weights with $\sum_{k=1}^n a_k = 1$, then $\phi(\sum_{k=1}^n a_k x_k) \leq \sum_{k=1}^n a_k \phi(x_k)$.

$\lambda_2 = \nu_1$, and the Lyapunov exponent itself yields an upper bound for the decay rate, that is, $\alpha \leq \Lambda$.

Next we will address the question to which extent this simple reasoning can be generalised and made rigorous for a larger class of systems.

2.2. Piecewise linear Markov maps

Focussing on piecewise linear Markov maps considerably reduces the need to worry about certain subtleties, as the transfer operator is compact and admits finite-dimensional matrix representations when considered on a space of piecewise analytic observables. Thus, at a computational level all technical details reduce to straightforward matrix manipulations.

The argument requires defining a suitable function space, on which the generalised transfer operator (see below) is compact. Results of this type for general analytic Markov maps are well known (see, for example, [**72**] or [**56**]). The special case of piecewise linear Markov maps, where a complete determination of the spectrum is possible, is folklore. We provide the details for the sake of completeness.

Definition 2.2.1. An interval map $T\colon I \to I$ is said to be a *Markov map* if there exists a finite partition $\{I_1, \ldots, I_K\}$ of I such that for any pair (k,l) either $T(\mathrm{int}(I_k)) \cap \mathrm{int}(I_l) = \emptyset$ or $\mathrm{int}(I_l) \subseteq T(\mathrm{int}(I_k))$. If this is the case, the corresponding partition will be referred to as a *Markov partition* and the $K \times K$ matrix A given by

$$A_{kl} = \begin{cases} 1 & \text{if } \mathrm{int}(I_l) \subseteq T(\mathrm{int}(I_k)), \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

will be called the *topological transition matrix* of the Markov map T.

A Markov map T with Markov partition $\{I_1, \ldots, I_K\}$ is said to be *expanding* if $|T'(x)| > 1$ for all $x \in \mathrm{int}(I_k)$. It is said to be *piecewise linear* if T' is constant on each element of the Markov partition, that is, $T'(x) = \gamma_k$ for all $x \in \mathrm{int}(I_k)$.

Finally, we call an expanding Markov map with topological transition matrix A *topologically mixing*[2] if there is a positive integer p such that each entry of the matrix A^p is strictly positive.

The main tool to establish the desired inequality (2.3) consists in studying the spectral properties of the generalised transfer operator (also referred to as Ruelle-Perron-Frobenius operator).

Definition 2.2.2. To any piecewise linear Markov map T and $\beta \in \mathbb{R}$, one can associate the *generalised transfer operator*

$$(\mathcal{L}_\beta f)(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|^\beta}\,, \tag{2.5}$$

[2]This is a slight abuse of terminology, since its use is usually restricted to continuous maps. However, it serves the same purpose as in the continuous setup as it guarantees the existence of a spectral gap for the corresponding transfer operator (see Corollary 2.2.7).

which is a bounded operator on $L^1(X,m)$.

For $\beta = 1$ this expression reduces to the (usual) transfer operator defined in (1.6). As we shall see next, the operator $\mathcal{L}_\beta$ in (2.5) leaves invariant certain subspaces of $L^1(X,m)$, namely spaces of functions analytic on each of $\mathrm{int}(I_k)$. Any such function f can be identified with the K-tuple of analytic functions, each of which is the analytic extension of $f|_{\mathrm{int}(I_k)}$ for some k. In order to make this more precise, we require some more notation.

Definition 2.2.3. Let D denote an open disk in the complex plane. We use

$$\mathcal{H}(D) = \bigoplus_{k=1}^{K} H^\infty(D)$$

to denote the space of K-tuples $h = (h_1, \ldots, h_K)$ of bounded holomorphic functions on D (see Definition 1.4.1). This is a Banach space when equipped with the norm $\|h\|_{\mathcal{H}(D)} = \max\{\|h_k\|_{H^\infty(D)} : k = 1, \ldots, K\}$.

Let T be a piecewise linear expanding Markov map with the Markov partition $\{I_1, \ldots, I_K\}$ and topological transition matrix A. For each (k,l) with $A_{lk} \neq 0$ we denote by $\varphi_{kl} \colon I_k \to I_l$ the inverse branch of the Markov map from partition element I_k into the partition element I_l, as well as its analytic continuation to the complex plane. Observe that, since the map is expanding, all inverse branches are contractions. We can thus choose two concentric open disks D_r and D_R in $\mathbb{C}$ with radii $0 < r < R$ such that

$$\varphi_{kl}(D_R) \subset D_r \Subset D_R \quad \text{for all inverse branches } \varphi_{kl}\,. \tag{2.6}$$

It turns out that $\mathcal{H}(D_R)$ is a suitable space of observables for the transfer operator $\mathcal{L}_\beta$ associated to T in the sense of the following proposition.

Proposition 2.2.4. *Let T, A, φ_{kl} and D_R be given as above. Then, for any real β, the transfer operator $\mathcal{L}_\beta$ viewed as an operator from $\mathcal{H}(D_R)$ to itself is well defined and bounded, and can be written as*

$$\left(\mathcal{L}_\beta h\right)_k(z) = \sum_l A_{lk} |\varphi'_{kl}(z)|^\beta h_l(\varphi_{kl}(z))\,. \tag{2.7}$$

Proof. The representation (2.7) follows from a short calculation using the definition of $\mathcal{L}_\beta$ in (2.5). The K-tuple $h = (h_1, \ldots, h_K)$ corresponds to the piecewise analytic function f on I, meaning that h_l is an analytic extension of $f|_{\mathrm{int}(I_l)}$. If $x \in I_k$, then (2.5) can be written as

$$(\mathcal{L}_\beta f)(x) = \sum_{l=1}^{K} A_{lk} \frac{f(\varphi_{kl}(x))}{|T'(\varphi_{kl}(x))|^\beta}\,.$$

As φ_{kl} is the analytic extension of the inverse branch which maps I_k to I_l, and $f|_{\mathrm{int}(I_l)} = h_l|_{\mathrm{int}(I_l)}$, we get the desired expression for $(\mathcal{L}_\beta h)_k$, which is the analytic extension of $(\mathcal{L}_\beta f)|_{\mathrm{int}(I_k)}$.

Since $|\varphi_{kl}'(z)|^\beta = |\gamma_l|^{-\beta}$ is constant and the disk D_R satisfies (2.6), the operator maps $\mathcal{H}(D_R)$ to $\mathcal{H}(D_R)$. In order to see that $\mathcal{L}_\beta \colon \mathcal{H}(D_R) \to \mathcal{H}(D_R)$ is bounded observe that if $h \in \mathcal{H}(D_R)$ with $\|h\|_{\mathcal{H}(D_R)} \leq 1$, then

$$\|\mathcal{L}_\beta h\|_{\mathcal{H}(D_R)} = \max_k \|(\mathcal{L}_\beta h)_k\|_{H^\infty(D_R)} \leq \max_k \sum_l A_{lk}|\gamma_l|^{-\beta} < \infty\,. \qquad \square$$

REMARK 2.2.5. The space $\mathcal{H}(D_R)$ is not the only suitable space of observables. Restricting to the same disk of analyticity for each branch, however, simplifies notation. More general spaces are discussed in [**11**] and [**12**].

Next we will show that $\mathcal{L}_\beta$ viewed as an operator on $\mathcal{H}(D_R)$ is compact. The proof relies on the factorisation argument explained in Section 1.4.1. We shall write $\tilde{\mathcal{L}}_\beta \colon \mathcal{H}(D_r) \to \mathcal{H}(D_R)$ for the lifted operator also given by the functional expression (2.7), which is bounded by the same argument as for $\mathcal{L}_\beta$. Further, $\mathcal{J}$ is the bounded embedding operator which maps $\mathcal{H}(D_R)$ injectively to $\mathcal{H}(D_r)$. To be precise, $\mathcal{J} \colon \mathcal{H}(D_R) \to \mathcal{H}(D_r)$ is given by $(\mathcal{J}h)_k = Jh_k$, where $J \colon H^\infty(D_R) \to H^\infty(D_r)$ in turn is given by $(Jh)(z) = h(z)$ for $z \in D_r$. As J is compact by Montel's theorem (see [**23**, Thm. 2.9, Ch. 7]) it follows that $\mathcal{J}$ is compact, and since $\tilde{\mathcal{L}}_\beta$ is bounded, the factorisation $\mathcal{L}_\beta = \tilde{\mathcal{L}}_\beta \mathcal{J}$ as in (1.18) implies that $\mathcal{L}_\beta$ is compact.

We now turn to the approximation argument in order to show that the spectrum of $\mathcal{L}_\beta$ is given by the eigenvalues of certain block matrices. For piecewise linear Markov maps the transfer operator is easily seen to map piecewise polynomial functions of degree at most N to piecewise polynomial functions of degree at most N. This follows from a straightforward calculation using the fact that the inverse branches are affine functions. Hence, in the natural basis of piecewise monomials, the operator $\mathcal{L}_\beta$ can be represented by the $(N+1)K \times (N+1)K$ block upper triangular matrix. To be precise, let $P_N \colon H^\infty(D_R) \to H^\infty(D_R)$ be the projection given in (1.21), and choose $\{e_n : n = 0, \dots, N\}$ with $e_n(z) = z^n$ as a basis for

$$H_N = P_N(H^\infty(D_R)).$$

For $\mathcal{H}_N = \bigoplus_{k=1}^K H_N$ choose the basis

$$\{\mathcal{E}_0^{(1)}, \dots, \mathcal{E}_0^{(K)}, \mathcal{E}_1^{(1)}, \dots, \mathcal{E}_1^{(K)}, \dots, \mathcal{E}_N^{(1)}, \dots, \mathcal{E}_N^{(K)}\},$$

where $\mathcal{E}_n^{(k)} = (0, \dots, 0, e_n, 0, \dots, 0)$ with e_n in the k-th position.

On $\mathcal{H}(D_R)$ we introduce the projection operator $\mathcal{P}_N \colon \mathcal{H}(D_R) \to \mathcal{H}(D_R)$ by setting $\mathcal{P}_N h = (P_N h_1, \dots, P_N h_K)$; note that $\mathcal{H}_N = \mathcal{P}_N(\mathcal{H}(D_R))$. Then, the restriction of $\mathcal{P}_N \mathcal{L} \mathcal{P}_N$ to $\mathcal{H}_N$ is represented by the $(N+1)K \times (N+1)K$ block upper triangular matrix

$$\begin{pmatrix} L^{(00)}(\beta) & L^{(01)}(\beta) & \cdots & L^{(0N)}(\beta) \\ 0 & L^{(11)}(\beta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & L^{(N-1N)}(\beta) \\ 0 & \cdots & 0 & L^{(NN)}(\beta) \end{pmatrix}. \qquad (2.8)$$

A simple calculation of the matrix representation $L^{(00)}(1)$ of $\mathcal{L}_1$ on the space of piecewise constant functions can be found in, for example, [**17**, p. 176]. More generally, it follows from (2.7) that the matrix elements of the block matrices $L^{(mn)}(\beta)$ are given in terms of the slopes γ_k and intercepts d_k of the branches of T, and the topological transition matrix $(A_{kl})_{1\leq k,l\leq K}$ induced by T and $\{I_1,\dots,I_K\}$ as follows[3]:

$$L^{(mn)}_{kl}(\beta) = \frac{A_{lk}}{|\gamma_l|^\beta \gamma_l^n}\cdot(-d_l)^{n-m}\binom{n}{m}. \tag{2.9}$$

The eigenvalues are determined by the diagonal blocks $L^{(mm)}(\beta)$ with matrix elements given by the first factor in (2.9).

We are now able to combine the factorisation with the approximation result above to prove the following.

Proposition 2.2.6. *Let T, A, φ_{kl} and D_R be as above. Then, for any real β, the transfer operator $\mathcal{L}_\beta$ viewed as an operator on $\mathcal{H}(D_R)$ is compact and its nonzero eigenvalues are precisely the nonzero eigenvalues of the matrices $L^{(mm)}(\beta)$ given in (2.9) with $m\in\mathbb{N}_0$, with the same multiplicities*[4].

Proof. By Lemma 1.4.7, we have $\lim_{N\to\infty}\|J-JP_N\|_{H^\infty(D_R)\to H^\infty(D_r)}=0$, which immediately leads to

$$\lim_{N\to\infty}\|\mathcal{J}-\mathcal{J}\mathcal{P}_N\|_{\mathcal{H}(D_R)\to\mathcal{H}(D_r)}=0\,.$$

As $\mathcal{L}_\beta(\mathcal{H}_N)\subseteq\mathcal{H}_N$, by Lemma 1.4.6 we have that the nonzero eigenvalues of $\mathcal{L}_\beta$ are precisely those of $\mathcal{P}_N\mathcal{L}_\beta\mathcal{P}_N$ for $N\in\mathbb{N}_0$, which are exactly the nonzero eigenvalues of the block matrices in (2.8). The assertion concerning multiplicities follows from [**27**, XI.9.5]. $\square$

Specialising to topologically mixing Markov maps, the Perron-Frobenius theorem (see, for example, [**32**, p. 53] or [**45**, p. 536]) guarantees that for any even m, the nonnegative[5] matrix $L^{(mm)}(\beta)$ in (2.9) has a simple, positive eigenvalue $\nu_m(\beta)$, referred to as the Perron eigenvalue of $L^{(mm)}(\beta)$, which is larger (in modulus) than all other eigenvalues. As a consequence, we obtain the following refinement of the above proposition.

Corollary 2.2.7. *Suppose that the hypotheses of the previous proposition hold. If the Markov map T is also topologically mixing, then $\mathcal{L}_\beta\colon\mathcal{H}(D_R)\to\mathcal{H}(D_R)$ has a simple positive leading eigenvalue $\nu_0(\beta)$. Moreover, this eigenvalue is the Perron eigenvalue of the matrix $L^{(00)}(\beta)$.*

[3]Note that $\varphi_{kl}(z)=(z-d_l)/\gamma_l$. Then $h=\mathcal{E}^{(l)}_n$ in (2.7) yields

$$(\mathcal{L}_\beta\mathcal{E}^{(l)}_n)_k(z)=A_{lk}|\varphi'_{kl}(z)|^\beta(\varphi_{kl}(z))^n=A_{lk}\frac{1}{|\gamma_l|^\beta}\frac{1}{\gamma_l^n}\sum_{i=0}^n\binom{n}{i}z^n(-d_l)^{n-i}\,.$$

The coefficient of z^m yields (2.9).

[4]This means, for any nonzero eigenvalue λ of $\mathcal{L}_\beta$, its multiplicity coincides with the sum of the multiplicities of λ as an eigenvalue of $L^{(11)}(\beta),\dots,L^{(NN)}(\beta)$ for large enough N.

[5]The matrix $L^{(mm)}(\beta)$ is not necessarily nonnegative for odd m.

PROOF. This follows from the previous proposition together with the observation that for $m \geq 1$ the spectral radius $\rho(L^{(mm)}(\beta))$ of the matrix $L^{(mm)}(\beta)$ is strictly smaller than the Perron eigenvalue of $L^{(00)}(\beta)$. In order to see this note that by (2.9) for all $m \geq 1$ we have

$$|L_{kl}^{(mm)}(\beta)| \leq C L_{kl}^{(00)}(\beta)\,, \tag{2.10}$$

where

$$C = \max_{l} \frac{1}{|\gamma_l|} < 1\,.$$

As (2.10) implies $|(L^{(mm)}(\beta))_{kl}^{p}| \leq C^p (L^{(00)}(\beta))_{kl}^{p}$ for each $p \geq 0$ and $m \geq 1$, we have

$$\left\| \left(L^{(mm)}(\beta)\right)^p \right\|_F \leq C^p \left\| \left(L^{(00)}(\beta)\right)^p \right\|_F\,,$$

where $\|\cdot\|_F$ denotes the Frobenius norm. The spectral radius formula (A.1) now implies that

$$\rho(L^{(mm)}(\beta)) \leq C\nu_0(\beta)\,. \qquad \square$$

The leading (positive) eigenvalue $\nu_0(\beta)$ of $L^{(00)}(\beta)$ determines the so-called *topological pressure*, given by $P(\beta) = \ln \nu_0(\beta)$ for topologically mixing T. The following lemma summarises the well-known properties of the pressure (see Figure 2.2), which have been established for a large class of dynamical systems (see, for example, [**41**, Ch. 4]). In the context of piecewise linear Markov maps, these can be deduced from the matrix representation using elementary methods.

LEMMA 2.2.8. *The topological pressure $P(\beta)$ has the following properties:*

(i) $P(1) = 0$*;*
(ii) P *is convex in* β*;*
(iii) P *is analytic in* β*;*
(iv) $\frac{\partial P}{\partial \beta}\big|_{\beta=1} = -\Lambda$*, where* Λ *is the Lyapunov exponent in* (1.2) *with respect to the unique acip measure* μ*.*

PROOF. Statement (i) follows immediately. To simplify notation for the other statements, we denote the first block matrix $L^{(00)}(\beta)$ in (2.8) by $L(\beta)$ for the remainder of the proof.

(ii) As $\nu_0(\beta)$ is a simple leading eigenvalue of $L(\beta)$, we have $P(\beta) = \ln \nu_0(\beta) = \lim_{n\to\infty}(1/n) \ln \operatorname{Tr}\left((L(\beta))^n\right)$, where Tr denotes the trace of a matrix. Convexity of the pressure will follow from the logarithmic convexity of the trace. A function $f\colon \mathbb{R} \to \mathbb{R}_+$ is called *logarithmically convex* if $\ln f$ is convex, or equivalently if $f(tx + (1-t)y) \leq f(x)^t f(y)^{1-t}$ for all $x, y \in \mathbb{R}$ and $t \in [0,1]$. Using Hölder's inequality, it is easy to see that the sum of two logarithmically convex functions is again logarithmically convex. Clearly, the same holds for the product of logarithmically convex functions. Now, since every matrix entry $L_{kl}(\beta)$ in (2.9) is logarithmically convex, so is $\operatorname{Tr}\left((L(\beta))^n\right)$. Thus the topological pressure $P(\beta)$ is convex.

(iii) As the leading eigenvalue $\nu_0(\beta)$ is simple for any β, and L is analytic in β, it follows that ν_0 and hence P are analytic in β, see [**39**, Thm. 1.8, Ch. II].

(iv) By (i) we have $\frac{\partial P}{\partial\beta}\big|_{\beta=1} = \frac{\partial\nu_0}{\partial\beta}\big|_{\beta=1}$. For any $\beta \in \mathbb{R}$, the matrix $L(\beta)$ has right and left eigenvectors $\rho(\beta)$ and $m(\beta)$ corresponding to the simple eigenvalue $\nu_0(\beta)$ satisfying $m(\beta)^T\rho(\beta) = 1$. The eigenvalue $\nu_0(\beta)$ and the eigenvectors $\rho(\beta)$ and $m(\beta)$ are analytic, see [**39**, Thm. 1.8, Ch. II]. For brevity, we write $\rho = \rho(1)$ and $m = m(1)$, and write ρ_k and m_k for the k-th entries of ρ and m, respectively. Observing that $m_k = |I_k|$ and $|\gamma_l||I_l| = \sum_{k=1}^K A_{lk}|I_k|$ and using $\left(\frac{\partial L}{\partial\beta}\right)_{kl} = \frac{A_{lk}}{|\gamma_l|^\beta}\ln\frac{1}{|\gamma_l|}$ we have

$$\frac{\partial\nu_0}{\partial\beta}\bigg|_{\beta=1} = m\left(\frac{\partial L}{\partial\beta}\bigg|_{\beta=1}\right)\rho = \sum_{l=1}^K\left(\sum_{k=1}^K\frac{A_{lk}|I_k|}{|\gamma_l|}\right)\ln\frac{1}{|\gamma_l|}\rho_l = -\sum_{l=1}^K|I_l|\ln|\gamma_l|\rho_l\,,$$

which finishes the proof as $\Lambda = \sum_{l=1}^K|I_l|\ln|\gamma_l|\rho_l$. □

Now, using this lemma we are able to establish a relation between the exponential mixing rate (see (1.10) and (1.14)) determined by the eigenvalues of the operator $\mathcal{L}_\beta$ for $\beta = 1$ and the Lyapunov exponent Λ in (1.2) with respect to the unique acip measure with corresponding piecewise constant density.

PROPOSITION 2.2.9. *Let $T\colon I \to I$ be a topologically mixing piecewise linear expanding Markov map, and D_R as above. For $\mathcal{H} = \mathcal{H}(D_R)$ as in Definition 2.2.3, the corresponding mixing rate $\alpha_\mathcal{H}$ is bounded in terms of the Lyapunov exponent:*

$$\alpha_\mathcal{H} \le 2\Lambda\,. \tag{2.11}$$

If all slopes γ_k have the same sign a sharper estimate holds:

$$\alpha_\mathcal{H} \le \Lambda. \tag{2.12}$$

PROOF. Recall that $\nu_m(\beta)$ denotes the largest eigenvalue (in modulus) of $L^{(mm)}(\beta)$. The largest eigenvalue of $\mathcal{L}_\beta$ for $\beta = 1$ is given by $\nu_0(1) = 1$, while $\nu_2(1)$ is a positive eigenvalue which provides a lower bound for the subleading eigenvalue of $\mathcal{L}_\beta$. Thus

$$\alpha_\mathcal{H} \le -\ln\nu_2(1)\,. \tag{2.13}$$

On the other hand, $L^{(22)}(\beta) = L^{(00)}(\beta+2)$ by (2.9), which implies

$$\nu_2(\beta) = \nu_0(\beta+2)\,. \tag{2.14}$$

Hence, using the properties of the topological pressure in Lemma 2.2.8, the relations (2.13) and (2.14) yield

$$\alpha_\mathcal{H} \le -P(3) \le (3-1)\Lambda\,.$$

See Figure 2.2 for a graphical illustration of the second inequality.

Note that if all slopes γ_k of T have the same sign, then we have $L^{(11)}(\beta) = \operatorname{sign}(\gamma_k)L^{(00)}(\beta+1)$. Thus, we can apply the previous arguments to $|\nu_1(1)|$ to obtain the following improved estimate:

$$\alpha_\mathcal{H} \le -\ln|\nu_1(1)| = -\ln\nu_0(2) = -P(2) \le (2-1)\Lambda\,.$$ □

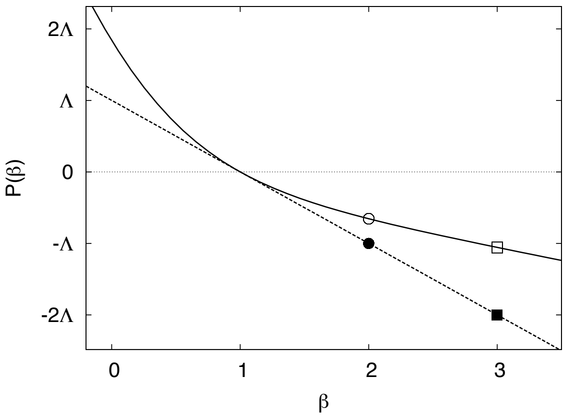

FIGURE 2.2. Schematic representation of the topological pressure $P(\beta)$ and its tangent at $\beta = 1$ with slope $-\Lambda$. The marked points are those used in the proof of Proposition 2.2.9.

REMARK 2.2.10. The assumption that T is topologically mixing is sufficient but not necessary. Indeed, there exist piecewise linear expanding Markov maps T with the following properties: the map T is not topologically mixing, yet exhibits exponential decay of correlations, that is $\alpha_{\mathcal{H}} > 0$, while the conclusions of Proposition 2.2.9 hold. One example is the piecewise linear Markov map on the interval $[0, 1]$ with four branches given by $T(x) = 2x + (1 - n)/2$ for $x \in [n/4, (n + 1)/4)$ and $n = 0, \ldots, 3$.

REMARK 2.2.11. The estimates (2.11) and (2.12) are sharp with simple examples achieving these bounds: the tent map ($\alpha_{\mathcal{H}} = 2\Lambda$) and the doubling map ($\alpha_{\mathcal{H}} = \Lambda$).

2.3. Remarks on nonlinear expanding maps

The setup of piecewise linear Markov maps is rather special. One may thus be tempted to ask whether a result like Proposition 2.2.9 extends, say, to Markov maps with nonlinear branches. While the previous considerations are based on the finite-matrix representation and on the eigenvalue relation (2.14), there is no obvious approach to the nonlinear case.

In order to get an idea how the nonlinearity affects relations (2.11) and (2.12), we approximate the spectrum of the transfer operator numerically. For this, we consider a simple example, a family of full branch piecewise Möbius maps F_c defined on $[-1, 1]$,

$$F_c(x) = \frac{1 - 2(c+1)|x|}{1 + 2c|x|} \,. \tag{2.15}$$

We restrict the parameter to $c \in (-1/4, 1/2)$, in order to guarantee expansivity. Figure 2.3(a) depicts the map F_c for $c = -0.22$.

There is an extensive body of literature on numerical approximation schemes for spectra of expanding systems, which broadly fall into two categories: those with a dynamical flavour based on cycle expansions (see [**4**]) of the Fredholm determinant (see [**20, 37**]), and those with a functional analytic flavour based on finite-rank approximations of transfer operators (see [**8, 31, 52**]).

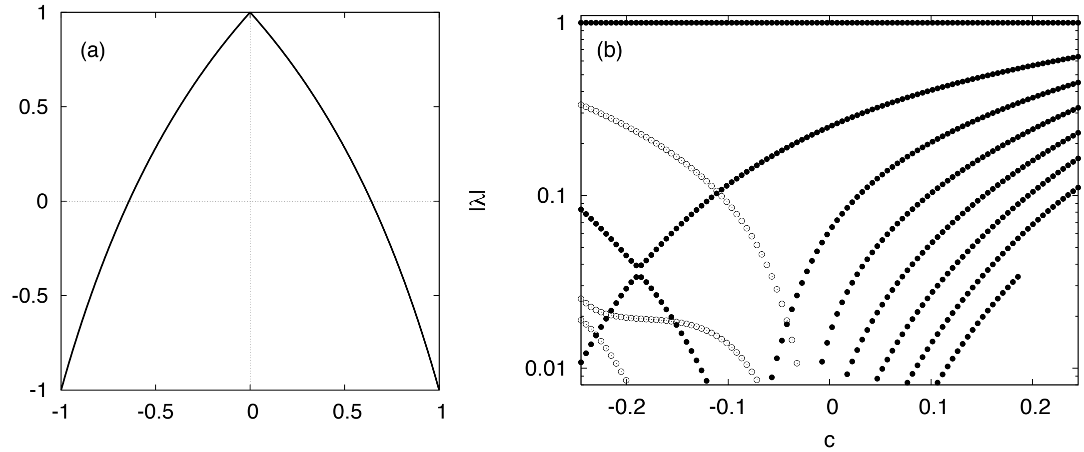

FIGURE 2.3. Left: (a) Diagrammatic view of the Möbius map (2.15) for $c < 0$. Right: (b) The largest eigenvalues (in modulus) of $\mathcal{L}_\beta$ for $\beta = 1$ for the map F_c in dependence on c, obtained using the Lagrange-Chebyshev method with truncation order $n = 25$. Positive/negative eigenvalues are indicated by filled/open symbols.

Following the spirit of the latter approach, we will effectively approximate the spectrum of $\mathcal{L}_\beta$ associated to F_c, considered on $H^\infty(U)$ for certain bounded domains $U \supset [-1, 1]$, using Lagrange-Chebyshev approximation. The basic idea of this method is to approximate $\mathcal{L}_\beta$ by an $n \times n$ matrix $P_n \mathcal{L}_\beta P_n$, where P_n denotes the projection that sends $f \in H^\infty(U)$ to its Lagrange-Chebyshev interpolating polynomial of degree $n-1$. This method is easily implemented and, moreover, it is possible to show (see [**10**]) that the eigenvalues of $P_n \mathcal{L}_\beta P_n$ converge exponentially fast to the eigenvalues of $\mathcal{L}_\beta$. This way the largest eigenvalues of $\mathcal{L}_\beta$ and their dependence on c are easily obtained (see Figure 2.3(b)). A minimum for the subleading eigenvalue occurs at about $c = -0.11$. The corresponding numerical eigenvalue reads $\lambda_2 \approx 0.10415$ resulting in a mixing rate $\alpha_{\mathcal{H}} = -\ln \lambda_2 \approx 2.2619$. The corresponding Lyapunov exponent is computed using the numerical approximation of the invariant density. The numerical value is $\Lambda \approx 0.685$ suggesting that the inequality (2.11) is violated.

However, by considering the transfer operator of F_c (or more generally any expanding piecewise smooth map $T\colon I \to I$ with, say, $I = [-1, 1]$, mixing with respect to its unique acip measure) on a larger function space (including discontinuous functions), the presence of nontrivial essential spectrum allows to establish an estimate like (2.12). For that purpose let us consider the transfer operator $\mathcal{L}$ on the space of functions of bounded variation[6] BV. In this setup, the spectrum of the transfer operator associated with expanding maps has been studied extensively. In particular,

[6]Recall that the total variation of a function $f\colon [-1,1] \to \mathbb{R}$ is defined as $\operatorname{var}(f) = \sup \big\{ \sum_{i=1}^{p} |f(x_i) - f(x_{i-1})| : -1 \le x_0 \le \cdots \le x_p \le 1 \big\}$. Then $f \in L^1(I, m)$ is in the space BV if there exists $\tilde{f}$ with $\tilde{f} = f$ a.e. such that $\operatorname{var}(\tilde{f}) < \infty$. This space is a Banach space with the norm given by

$$\|f\|_{BV} = \|f\|_1 + \inf_{\tilde{f}=f \text{ a.e.}} \operatorname{var}(\tilde{f}).$$

there is an explicit formula for the essential spectral radius [**40**]:

$$\rho_{ess} = \lim_{k\to\infty} (\inf\{|(T^k)'(x)| : x \in I\})^{-1/k} .$$

Thus, we have an upper bound for the mixing rate (Section 1.3)

$$\alpha_{BV} \leq -\ln \rho_{ess} = \lim_{k\to\infty} \frac{1}{k} \ln \inf\{|(T^k)'(x)| : x \in I\} ,$$

which yields the following estimate for the Lyapunov exponent

$$\begin{aligned} \Lambda &= \frac{1}{k} \int_I \ln |(T^k)'(x)| \, d\mu \\ &\geq \frac{1}{k} \inf\{\ln |(T^k)'(x)| : x \in I\} \\ &= \frac{1}{k} \ln \inf\{|(T^k)'(x)| : x \in I\} , \end{aligned}$$

for any $k \in \mathbb{N}$. For observables of bounded variation we get the following result.

PROPOSITION 2.3.1. *Let $T: I \to I$ be a piecewise smooth expanding interval map which is mixing with respect to its unique acip measure. Then the mixing rate on BV is bounded by the Lyapunov exponent,*

$$\alpha_{BV} \leq \Lambda .$$

In fact, almost identical statements can be found in [**21**], for example, Corollary 9.2.

There seems to be no simple answer to the question about the relation between Lyapunov exponents and mixing rates. Correlation decay depends crucially on properties of the observables, and it is a pivotal question which observables are physically relevant. Even if a real world phenomenon is sufficiently well modelled by a smooth dynamical system, one should keep in mind that modern digital data processing inevitably leads to discontinuous observations. Here, observables of bounded variation could be the relevant class for applications and in these cases Proposition 2.3.1 applies.

If one restricts to piecewise linear expanding Markov maps on an interval observed via piecewise analytic functions, then Proposition 2.2.9 establishes a relation between mixing rates and Lyapunov exponent. However, the above numerical simulations (see also [**20**]) suggest that the bound obtained in Propostion 2.2.9 cannot be simply extended to nonlinear expanding maps. The question remains whether a more general bound on the mixing rate α (that is, a lower bound on the subleading eigenvalue) in terms of the Lyapunov exponent Λ can be established.

Of these two quantities, the invariant density, and hence the Lyapunov exponent, seems to be generally more easily accessible. For some examples the invariant density is known, or alternatively a transformation with a given invariant density can sometimes be reverse engineered. The latter problem is known as the inverse Perron-Frobenius problem, and has been investigated in several one-dimensional settings, see [**26, 29, 34**].

In contrast to this, not a single nontrivial example is known for which the subleading eigenvalue and the corresponding eigenfunction of a (compact) transfer operator are known explicitly. To the best of the author's knowledge, the only one-dimensional examples for which the complete spectrum is analytically accessible are piecewise linear full branch interval maps (see, for example, [**33**]), or more generally, piecewise linear Markov interval maps (see Section 2.2 or [**61**]) and circle maps of the form $z \mapsto z^n$ for $n \in \mathbb{N}$.

In the next chapters we shall construct a class of (nonlinear) expanding interval and circle maps, for which we explicitly determine the entire spectrum of the associated transfer operator. This will ultimately allow us to show that a bound like in Proposition 2.2.9 cannot be generalised to all nonlinear expanding maps, by providing a family of interval maps for which Λ is bounded while α can be arbitrarily large.

CHAPTER 3

Explicit spectra for a family of analytic circle maps

Motivated by the questions in the previous section, in this chapter we shall construct nontrivial examples of dynamical systems for which the spectrum of the associated transfer operator can be determined explicitly.

In Section 3.1 we start by constructing a family of interval maps with given eigenfunction and eigenvalue for the associated transfer operator, and show that these give rise to analytic circle maps. To create a suitable setting, we shall define in Section 3.2 Banach spaces of holomorphic functions on annuli, on which the associated transfer operators for analytic expanding circle maps are compact, in analogy with Section 1.4. Then, in Section 3.3 we explicitly determine the entire spectrum for the circle maps constructed in Section 3.1. Interestingly, when considered on the interval, these maps provide counterexamples to a variant of Mayer's conjecture on the reality of spectra for transfer operators (see Section 3.4).

The results of this chapter are published as [**83**].

3.1. A family of maps

In order to obtain explicit spectral information in the setting of nonlinear maps, we will turn the classical question of finding eigenvalues of the transfer operator for a given map on its head, and attempt to construct a map for which the transfer operator has a given eigenvalue with a given eigenfunction. Considering an analytic expanding full branch map on the interval I with two branches (Definition 1.4.2), the eigenvalue problem of the transfer operator in (1.15) formally reads

$$\mu_n u_n = \Phi_1' \cdot (u_n \circ \Phi_1) + \Phi_2' \cdot (u_n \circ \Phi_2), \tag{3.1}$$

where μ_n and u_n for $n \in \mathbb{N}_0$ are eigenvalues and eigenfunctions of $\mathcal{L}$, and Φ_1, Φ_2 the two inverse branches with $\Phi_k' > 0$.

Given an analytic invariant density ϱ one may consider (3.1) for $n = 0$ with $\mu_0 = 1$ and $u_0 = \varrho$ as an equation to compute suitable inverse branches Φ_1 and Φ_2 of the map. This setup is a particularly simple case of the so-called inverse Perron-Frobenius problem [**29, 34**], which has been applied in various guises to tailor-make chaotic maps with given stationary properties (see, for example, [**26**]). As we are attempting to construct a map with two branches we are at liberty to specify a nontrivial eigenvalue and corresponding eigenfunction. Thus, given an invariant density ϱ, a real number

λ with $|\lambda| < 1$, and a potential eigenfunction u, we seek to solve

$$\begin{aligned} P &= P \circ \Phi_1 + P \circ \Phi_2, \\ \lambda U &= U \circ \Phi_1 + U \circ \Phi_2 \end{aligned} \tag{3.2}$$

for the inverse branches Φ_1 and Φ_2. Here P and U denote suitable antiderivatives of ϱ and u, respectively. A priori, there is no guarantee that (3.2) admits a real solution for Φ_1 and Φ_2 and that such a solution actually determines an analytic full branch interval map. Developing general conditions under which this is the case seems to be a challenging task. Nevertheless, if we fix the interval $I = [-1, 1]$, take ϱ to be the uniform density, and $u(x) = \cos(\pi x)$ the eigenfunction with eigenvalue $\lambda \in (-1, 1)$, then (3.2) leads to

$$\begin{aligned} x &= \Phi_1(x) + \Phi_2(x), \\ \lambda \sin(\pi x) &= \sin(\pi \Phi_1(x)) + \sin(\pi \Phi_2(x)). \end{aligned} \tag{3.3}$$

A short calculation[1] yields explicit expressions for Φ_1 and Φ_2:

$$\Phi_1(x) = \frac{x}{2} - \frac{1}{\pi} \arccos\left(\lambda \cos\left(\frac{\pi x}{2}\right)\right)$$

and

$$\Phi_2(x) = \frac{x}{2} + \frac{1}{\pi} \arccos\left(\lambda \cos\left(\frac{\pi x}{2}\right)\right).$$

For any $\lambda \in (-1, 1)$ these indeed determine an analytic full branch map (see Figure 3.1) with its associated transfer operator having λ as an eigenvalue with eigenfunction $x \mapsto \cos(\pi x)$ by construction.

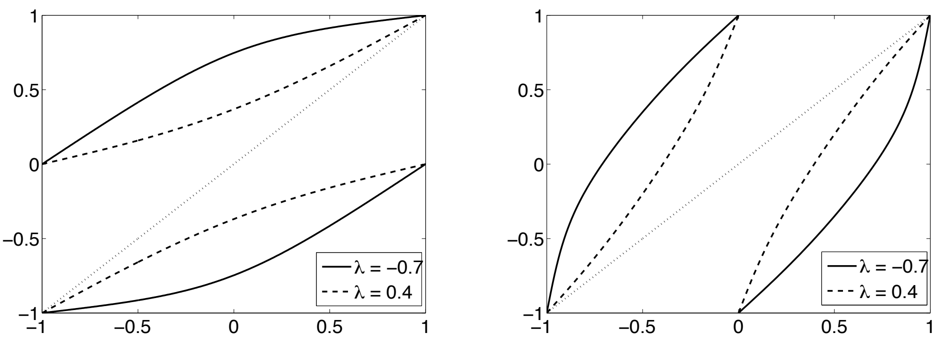

FIGURE 3.1. Inverse branches Φ_1 and Φ_2 (left) and the corresponding interval map (right) for $\lambda = -0.7$ and $\lambda = 0.4$.

It turns out that this particular example can be lifted to an analytic circle map, as Φ_1 and Φ_2 are analytic on $\mathbb{R}$ and $\Phi_2(x) = \Phi_1(x + 2)$, thus a solution to (3.3) lifts to $\Phi\colon \mathbb{R} \to \mathbb{R}$, where

$$\Phi(x) = \Phi_1(x).$$

[1]We use the identities $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ and $\sin(\alpha) + \sin(\beta) = 2\sin((\alpha+\beta)/2)\cos((\alpha-\beta)/2)$.

This is an increasing diffeomorphism with inverse $F\colon \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = 2x + 1 + \frac{2}{\pi} \arctan\left(\frac{\lambda \sin(\pi x)}{1 - \lambda \cos(\pi x)}\right). \tag{3.4}$$

Note that F is a lift of a circle map $\tau\colon \mathbb{T} \to \mathbb{T}$, which satisfies $F(x+2) = F(x) + 4$ and $p \circ F = \tau \circ p$, where $p\colon \mathbb{R} \to \mathbb{T}$ is the projection defined by $p(x) = e^{i\pi x}$. The map τ is a twofold covering of $\mathbb{T}$. Note that $F' > 1$ for $\lambda \in (-1, 1)$. Thus τ is an analytic expanding circle map (Definition 3.2.1). It turns out that τ can be written in closed form as

$$\tau(z) = z\frac{\lambda - z}{1 - \lambda z} \quad \text{for } z \in \mathbb{T}\,. \tag{3.5}$$

This can be seen using the relation $e^{i\pi F(x)} = \tau(e^{i\pi x})$, from which it follows that

$$\pi F(x) = \arg(\tau(e^{i\pi x})) = \pi x + \arg\left(\frac{\lambda - e^{i\pi x}}{1 - \lambda e^{i\pi x}}\right) = 2\pi x + \pi + 2\arg(1 - \lambda e^{-i\pi x})\,.$$

As we shall see in Section 3.3, for the above family of analytic circle maps λ is the second largest eigenvalue (in modulus) of the associated transfer operator. Moreover, we will obtain its entire spectrum (Theorem 3.3.1), which consists of infinitely many distinct nontrivial eigenvalues. Before embarking on these results, we shall first discuss properties of analytic expanding circle maps and define Banach spaces of holomorphic functions on which the corresponding transfer operators are compact (similarly to the interval setting in Section 1.4).

REMARK 3.1.1. As we will discuss in Chapter 4, the expanding circle map τ in (3.5) belongs to the class of finite Blaschke products (see Defintion 4.3.1). It is well known [**81**] that any two expanding circle maps of the same degree are topologically conjugate. Moreover, if this conjugacy is absolutely continuous then the two maps are smoothly conjugate [**82**]. However, while the map τ is topologically conjugate to $z \mapsto z^2$, this conjugacy is not smooth (except for $\lambda = 0$) as the multipliers of their periodic points do not coincide.

3.2. Transfer operators for analytic circle maps

We start by defining the notion of analytic expanding circle map.

DEFINITION 3.2.1. We say that $\tau\colon \mathbb{T} \to \mathbb{T}$ is an *analytic expanding circle map* if the following two conditions hold:

(i) τ is analytic on $\mathbb{T}$;
(ii) $\inf_{z\in\mathbb{T}} |\tau'(z)| > 1$.

It is not difficult to see that τ is a K-fold covering of $\mathbb{T}$ for some integer $K > 1$. Moreover, the map τ has analytic extensions to certain annuli containing $\mathbb{T}$. With slight abuse of notation we shall write τ for the various extensions as well. To be precise, for $r < 1 < R$ let $A_{r,R}$ denote the open annulus $A_{r,R} = \{\, z \in \mathbb{C} \,:\, r < |z| < R \,\}$ and write

$$\mathcal{A} = \{A_{r,R} : \tau \text{ and } 1/\tau \text{ holomorphic on } A_{r,R}\}\,.$$

Expansivity of τ yields the following result.

LEMMA 3.2.2. *For an analytic expanding circle map τ, there is $A_0 \in \mathcal{A}$ such that*

(a) both τ and $1/\tau$ are analytic on the closure $\mathrm{cl}(A_0)$ of A_0;
(b) $\tau(\partial A_0) \cap \mathrm{cl}(A_0) = \emptyset$, where ∂A_0 denotes the boundary of A_0.

PROOF. Since τ is an analytic expanding circle map it is possible to choose $A_1 \in \mathcal{A}$ such that both τ and $1/\tau$ are analytic on $\mathrm{cl}(A_1)$ with

$$\alpha := \inf_{z \in A_1} \left|\tau'(z)\right| > 1\,.$$

It is not difficult to see that $(\rho, \theta) \mapsto \log\left|\tau(\rho e^{i\theta})\right|$ is differentiable for all (ρ,θ) with $\rho e^{i\theta} \in A_1$ and

$$\frac{\partial}{\partial \rho} \log\left|\tau(\rho e^{i\theta})\right| = \Re\left(e^{i\theta}\frac{\tau'(\rho e^{i\theta})}{\tau(\rho e^{i\theta})}\right)\,, \tag{3.6}$$

$$\frac{\partial}{\partial \theta} \log\left|\tau(\rho e^{i\theta})\right| = -\Im\left(\rho e^{i\theta}\frac{\tau'(\rho e^{i\theta})}{\tau(\rho e^{i\theta})}\right)\,, \tag{3.7}$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary part of $z \in \mathbb{C}$. Since τ leaves $\mathbb{T}$ invariant, equation (3.7) implies either

$$e^{i\theta}\frac{\tau'(e^{i\theta})}{\tau(e^{i\theta})} \geq \alpha \quad \text{for all } \theta \in \mathbb{R}\,, \tag{3.8}$$

or

$$e^{i\theta}\frac{\tau'(e^{i\theta})}{\tau(e^{i\theta})} \leq -\alpha \quad \text{for all } \theta \in \mathbb{R}\,. \tag{3.9}$$

Suppose now that (3.8) holds (the other case can be dealt with similarly). Fixing β with $1 < \beta < \alpha$ we can choose $A_{r,R} \in \mathcal{A}$ with $A_{r,R} \subset A_1$ and $e^{\beta(r-1)} < r$, $e^{\beta(R-1)} > R$ such that

$$\Re\left(e^{i\theta}\frac{\tau'(\rho e^{i\theta})}{\tau(\rho e^{i\theta})}\right) \geq \beta \quad \text{for all } \rho \in [r,R] \text{ and } \theta \in \mathbb{R}\,.$$

Equation (3.6) now implies

$$\log\left|\tau(e^{i\theta})\right| - \log\left|\tau(re^{i\theta})\right| = \Re\int_r^1 e^{i\theta}\frac{\tau'(\rho e^{i\theta})}{\tau(\rho e^{i\theta})}\,d\rho \geq \beta(1-r)$$

and

$$\log\left|\tau(Re^{i\theta})\right| - \log\left|\tau(e^{i\theta})\right| = \Re\int_1^R e^{i\theta}\frac{\tau'(\rho e^{i\theta})}{\tau(\rho e^{i\theta})}\,d\rho \geq \beta(R-1)\,.$$

Thus

$$\left|\tau(re^{i\theta})\right| \leq e^{\beta(r-1)} < r \qquad \text{and} \qquad \left|\tau(Re^{i\theta})\right| \geq e^{\beta(R-1)} > R\,,$$

so $A_0 := A_{r,R}$ has all the desired properties. □

Given an expanding circle map τ, we associate with it (using Definition 1.3.1) a transfer operator $\mathcal{L}$ which is well-defined and bounded as an operator on $L^1(\mathbb{T}) = L^1(\mathbb{T}, m)$, where $m = d\theta/2\pi$ is the normalised one-dimensional Lebesgue measure on $\mathbb{T}$. Let ϕ_k denote the k-th local inverse branch[2] of the K-fold covering τ, then $\mathcal{L}\colon L^1(\mathbb{T}) \to L^1(\mathbb{T})$ for the circle map τ is given by

$$(\mathcal{L}f)(z) = \sum_{k=1}^{K} \phi_k'(z)(f \circ \phi_k)(z) \quad (a.e.\ z \in \mathbb{T})\,, \tag{3.10}$$

as for any $f \in L^1(\mathbb{T})$ and any $g \in L^\infty(\mathbb{T})$ we have

$$\begin{aligned} \frac{1}{2\pi i}\int_{\mathbb{T}} (\mathcal{L}f)(z)\cdot g(z)\,dz &= \frac{1}{2\pi i}\int_{\mathbb{T}} f(z)\cdot (g\circ\tau)(z)\,dz \\ &= \sum_{k=1}^{K} \frac{1}{2\pi i}\int_{\phi_k(\mathbb{T})} f(z)\cdot (g\circ\tau)(z)\,dz \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \left(\sum_{k=1}^{K} \phi_k'(z)\cdot (f\circ\phi_k)(z)\right)\cdot g(z)\,dz, \end{aligned} \tag{3.11}$$

where we used a change of variables and the fact that $\bigcup_{k=1}^{K}\phi_k(\mathbb{T}) = \mathbb{T}$ up to a set of measure zero.

It turns out that for suitable domains U containing $\mathbb{T}$, the operator $\mathcal{L}$ leaves $H^\infty(U)$ invariant. The proof will rely on Fourier theory. Here and in the following, we shall use

$$c_n(f) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{z^{n+1}}\,dz \quad (n \in \mathbb{Z}) \tag{3.12}$$

to denote the n-th Fourier coefficient of $f \in L^1(\mathbb{T})$. Then, using the definition of $\mathcal{L}$ (first equality in (3.11)) we can express the n-th Fourier coefficient of $\mathcal{L}f$ as

$$c_n(\mathcal{L}f) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{\tau(z)^{n+1}}\,dz \quad (n \in \mathbb{Z})\,. \tag{3.13}$$

A result[3] analogous to Lemma 1.4.4 for the case of expanding circle maps now reads as follows.

Lemma 3.2.3. *Suppose that annuli A, A' and A_0 in $\mathcal{A}$ are chosen[4] such that*

$$A_0 \Subset A' \subset A \text{ and } \tau(\partial A_0) \cap \mathrm{cl}(A) = \emptyset\,. \tag{3.14}$$

Then the transfer operator $\mathcal{L}$ maps $H^\infty(A')$ continuously to $H^\infty(A)$.

Proof. Given $f \in H^\infty(A')$, we shall show that $\mathcal{L}f \in H^\infty(A)$ by estimating the asymptotic behaviour of the Fourier coefficients of $\mathcal{L}f$. We denote by $\mathbb{T}_\rho = \{z \in \mathbb{C} : |z| = \rho\}$ the circle of radius ρ centred at 0. Write R_0 and R to denote the radii of the circles $\mathbb{T}_{R_0}$ and $\mathbb{T}_R$ forming the 'exterior' boundary of A_0 and A, respectively (see

[2]We use lower case Greek letters to denote inverse branches of circle maps and the corresponding upper case letters to denote inverse branches of interval maps.

[3]Note, that we can not simply use Lemma 1.4.4 as ϕ_k is not analytic on $\mathbb{T}$.

[4]This is always possible by Lemma 3.2.2.

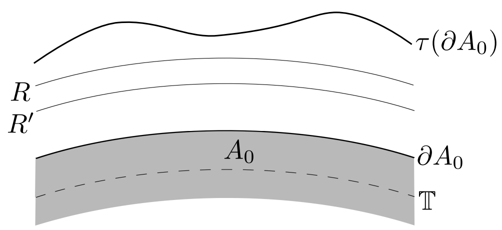

FIGURE 3.2. Proof of Lemma 3.2.3: choice of annuli A_0, A' and A.

Figure 3.2). Next choose[5] R'' with

$$\inf_{z\in\mathbb{T}_{R_0}} |\tau(z)| > R'' > R\,.$$

Similarly, write r_0 and r to denote the radii of the circles forming the 'interior' boundary of A_0 and A, respectively, and choose r'' with

$$\sup_{z\in\mathbb{T}_{r_0}} |\tau(z)| < r'' < r\,.$$

Fix $f \in H^\infty(A')$ with $\|f\|_{H^\infty(A')} \leq 1$ and let $n \geq 0$. Using (3.13) we see that

$$\begin{aligned}|c_n(\mathcal{L}f)| = \left|\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{\tau(z)^{n+1}}\,dz\right| &= \left|\frac{1}{2\pi i}\int_{\mathbb{T}_{R_0}} \frac{f(z)}{\tau(z)^{n+1}}\,dz\right| \\ &\leq \frac{1}{2\pi}\int_{\mathbb{T}_{R_0}} \frac{1}{|\tau(z)|^{n+1}}\,|dz| \leq \frac{R_0}{(R'')^{n+1}}\,.\end{aligned}$$

Similarly, for $n \geq 1$ we have

$$\begin{aligned}|c_{-n}(\mathcal{L}f)| = \left|\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{\tau(z)^{-n+1}}\,dz\right| &= \left|\frac{1}{2\pi i}\int_{\mathbb{T}_{r_0}} f(z)\tau(z)^{n-1}\,dz\right| \\ &\leq \frac{1}{2\pi}\int_{\mathbb{T}_{r_0}} |\tau(z)|^{n-1}\,|dz| \leq r_0(r'')^{n-1}\,.\end{aligned}$$

Hence, $\sum_{n=0}^\infty c_n(\mathcal{L}f)z^n$ converges absolutely for all $|z| \leq R$ and $\sum_{n=1}^\infty c_{-n}(\mathcal{L}f)z^{-n}$ converges absolutely for all $|z| \geq r$. Moreover, for $z \in A$ we have

$$\begin{aligned}\left|\sum_{n=-\infty}^{\infty} c_n(\mathcal{L}f)z^n\right| &\leq \sum_{n=0}^\infty |c_n(\mathcal{L}f)|\,R^n + \sum_{n=1}^\infty |c_{-n}(\mathcal{L}f)|\,r^{-n} \\ &\leq \sum_{n=0}^\infty \frac{R_0R^n}{(R'')^{n+1}} + \sum_{n=1}^\infty \frac{r_0(r'')^{n-1}}{r^n} = \frac{R_0}{R''-R} + \frac{r_0}{r-r''}\,.\end{aligned}$$

[5]Here, we assume that τ is orientation-preserving. The orientation-reversing case is similar.

Thus, by the uniqueness of the Fourier transform on $L^1(\mathbb{T})$, we conclude that $\mathcal{L}f \in H^\infty(A)$ and

$$\|\mathcal{L}f\|_{H^\infty(A)} \le \left(\frac{R_0}{R''-R} + \frac{r_0}{r-r''}\right)\|f\|_{H^\infty(A')} \quad \text{for all } f \in H^\infty(A')\,. \qquad \square$$

Choosing $A = A'$ in the previous lemma shows that $\mathcal{L}$ induces a well-defined continuous operator from $H^\infty(A)$ to itself.

Moreover, the operator $\mathcal{L}\colon H^\infty(A) \to H^\infty(A)$ is compact, which follows from a factorisation argument (see also Section 1.4.1). Given $A, A' \in \mathcal{A}$ with $A' \Subset A$ define the canonical embedding $J\colon H^\infty(A) \to H^\infty(A')$ by

$$Jf = f|_{A'}\,. \tag{3.15}$$

The embedding J is compact by Montel's theorem [**23**, Thm. 2.9, Ch. 7].

Now, for the choice of $A' \Subset A$ satisfying (3.14), the transfer operator $\mathcal{L}\colon H^\infty(A) \to H^\infty(A)$ factorises as

$$\mathcal{L} = \tilde{\mathcal{L}}J, \tag{3.16}$$

where $\tilde{\mathcal{L}}$ is the transfer operator viewed as an operator from $H^\infty(A')$ to $H^\infty(A)$, guaranteed to be continuous by Lemma 3.2.3. Thus, the factorisation (3.16) implies the following result.

Proposition 3.2.4. *Let $A \in \mathcal{A}$ with $A_0 \Subset A$. Then $\mathcal{L}\colon H^\infty(A) \to H^\infty(A)$ is compact.*

For future use we shall now show that J is well approximated by the following operators: for N a positive integer, define the finite rank operator $J_N\colon H^\infty(A) \to H^\infty(A')$ by

$$(J_N f)(z) = \sum_{n=-N-1}^{N-1} c_n(f) z^n \quad \text{for } z \in A'\,. \tag{3.17}$$

Lemma 3.2.5. *Let J and J_N be defined as above. Then*

$$\lim_{N\to\infty} \|J - J_N\|_{H^\infty(A)\to H^\infty(A')} = 0\,.$$

In particular, the embedding J is compact.

Proof. Choose $A'' \in \mathcal{A}$ with

$$A' \Subset A'' \Subset A\,.$$

Let R' and r' denote the radii of the circles forming the 'exterior' and 'interior' boundary of A', and analogously for R'' and r'' so that

$$r'' < r' < R' < R''\,.$$

Fix $f \in H^\infty(A)$ with $\|f\|_{H^\infty(A)} \le 1$. Then

$$\begin{aligned}\|Jf - J_N f\|_{H^\infty(A')} &= \sup_{z\in A'} \left| \sum_{n\ge N} \frac{z^n}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta + \sum_{n\ge N+2} \frac{z^{-n}}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta^{-n+1}}\, d\zeta \right| \\ &\le \sum_{n\ge N} \frac{(R')^n}{2\pi} \int_{\mathbb{T}_{R''}} \frac{|f(\zeta)|}{|\zeta|^{n+1}}\, |d\zeta| + \sum_{n\ge N+2} \frac{(r')^{-n}}{2\pi} \int_{\mathbb{T}_{r''}} \frac{|f(\zeta)|}{|\zeta|^{-n+1}}\, |d\zeta| \\ &\le \left(\frac{R'}{R''}\right)^N \frac{1}{1-\frac{R'}{R''}} + \left(\frac{r''}{r'}\right)^{N+2} \frac{1}{1-\frac{r''}{r'}}\,,\end{aligned}$$

from which the assertions follow. □

3.3. Spectrum for a family of circle maps

In this section we consider the family of analytic expanding circle maps constructed in Section 3.1 and show that the spectrum of $\mathcal{L}$ can be determined explicitly, which is the statement of Theorem 3.3.1.

Note first that the expression for τ in (3.5) yields an analytic circle map not just for real λ, but for any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ (see Figure 3.3) if written as

$$\tau(z) = z\frac{\lambda - z}{1 - \bar{\lambda} z} \quad \text{for } z \in \mathbb{T}\,, \tag{3.18}$$

which is expanding since for $z \in \mathbb{T}$ we have

$$|\tau'(z)| = \left|\tau'(z)\frac{z}{\tau(z)}\right| = 1 + \frac{1-|\lambda|^2}{|\lambda - z|^2} > 1.$$

It is possible to write down lifts of (3.18) for complex λ similar to (3.4). In fact, a short calculation shows that if $\lambda = |\lambda| e^{i\alpha}$ then the argument of arctan in (3.4) needs to be replaced by $|\lambda|\sin(\pi x - \alpha)/(1 - |\lambda|\cos(\pi x - \alpha))$.

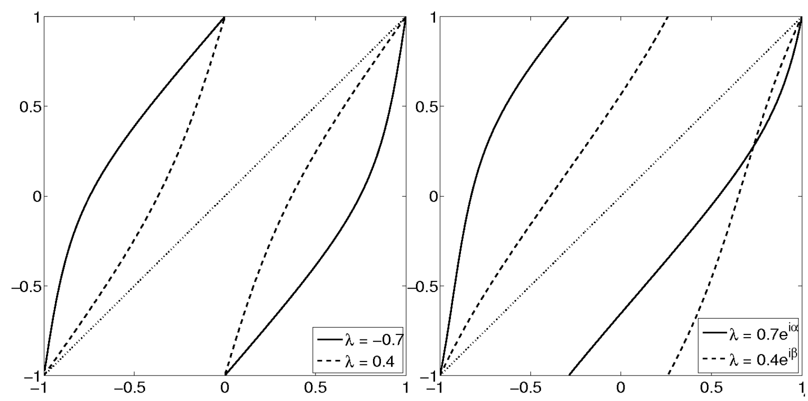

FIGURE 3.3. τ projected on the interval $[-1, 1]$ for (left) $\lambda = -0.7$ and $\lambda = 0.4$ and (right) $\lambda = -0.3 - i\sqrt{0.4} = 0.7e^{i\alpha}$ with $\alpha \approx -2.0137$ and $\lambda = 0.1 + i\sqrt{0.15} = 0.4e^{i\beta}$ with $\beta \approx 1.318$. Note that the projection is chosen such that the interval endpoint -1 is fixed by τ.

Given τ as in (3.18), we now choose an annulus $A \in \mathcal{A}$ with $A_0 \Subset A$. By Proposition 3.2.4 the associated transfer operator $\mathcal{L}\colon H^\infty(A) \to H^\infty(A)$ is well defined and compact. Moreover, all eigenvalues of $\mathcal{L}$ can be determined explicitly, which is the essence of the following theorem.

THEOREM 3.3.1. *For any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ the eigenvalues of the transfer operator $\mathcal{L}\colon H^\infty(A) \to H^\infty(A)$ associated to τ in (3.18) are precisely all nonnegative powers of λ and $\overline{\lambda}$, that is, the spectrum of $\mathcal{L}$ is*

$$\sigma(\mathcal{L}) = \{1\} \cup \{\, \lambda^n \,:\, n \in \mathbb{N}\,\} \cup \left\{ \overline{\lambda}^n \,:\, n \in \mathbb{N} \right\} \cup \{0\}\,.$$

Moreover, the algebraic multiplicity of the leading eigenvalue is 1, while the algebraic (and geometric) multiplicity of each λ^n is equal to its number of occurrences in the above list, meaning it is 1 if $\lambda^n \neq \overline{\lambda}^n$ and 2 otherwise.

The proof of this theorem relies on the fact that the spectrum of $\mathcal{L}$ can be computed by analysing the spectrum of a suitable matrix representation, which is obtained as follows. For $N \in \mathbb{N}$ consider the projection P_N given by the same functional expression as J_N in (3.17), now viewed as an operator from $H^\infty(A)$ to itself. Clearly, $P_N\mathcal{L}P_N$ is an operator of rank $2N+1$. Writing $e_n(z) = z^n$, the set $\{\, e_n \,:\, -N-1 \leq n \leq N-1\,\}$ is a basis for

$$H_N = P_N(H^\infty(A))$$

and the restriction of $P_N\mathcal{L}P_N$ to H_N is represented by the $(2N+1)\times(2N+1)$ matrix $L^{(N)}$ defined by

$$(L^{(N)})_{n,l} = c_{n-1}(\mathcal{L}e_{l-1}) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{z^l}{\tau(z)^n}\frac{dz}{z} = \frac{1}{2\pi i}\int_{\mathbb{T}} z^{l-n}\Big(\frac{1-\overline{\lambda}z}{\lambda - z}\Big)^n \frac{dz}{z}\,, \tag{3.19}$$

where we used (3.13) for the second equality.

In particular, the nonzero spectrum of $P_N\mathcal{L}P_N$ is given by the nonzero spectrum of $L^{(N)}$. Observe that (3.19) defines an infinite matrix L containing $L^{(N)}$ as a finite submatrix. The following lemma summarises the properties of L.

LEMMA 3.3.2. *For $l, n \in \mathbb{Z}$ the following hold:*

(a) $L_{0,0} = 1$;
(b) $L_{0,l} = 0$ *if* $l \neq 0$;
(c) $L_{-n,-l} = \overline{L_{n,l}}$;
(d) $L_{-n,-n} = \lambda^n$ *for* $n \geq 0$;
(e) $L_{n,l} = L_{-n,-l} = 0$ *for* $n > l$.

PROOF. Assertions (a) and (b) immediately follow from (3.19),
while (c) is a consequence of

$$\begin{aligned} L_{-n,-l} &= \frac{1}{2\pi i}\int_{\mathbb{T}} z^{n-l}\Big(\frac{\lambda - z}{1-\overline{\lambda}z}\Big)^n\frac{dz}{z} = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta(n-l)}\Big(\frac{\lambda - e^{i\theta}}{1-\overline{\lambda}e^{i\theta}}\Big)^n\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta(l-n)}\Big(\frac{1-\lambda e^{-i\theta}}{\overline{\lambda} - e^{-i\theta}}\Big)^n\,d\theta = \overline{L_{n,l}}\,. \end{aligned}$$

For (d) and (e), observe that $z \mapsto (\lambda - z)/(1 - \overline{\lambda} z)$ is holomorphic for all z in the closed unit disk. Thus, by the Residue theorem,

$$L_{-n,-n} = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{1}{z} \Big(\frac{\lambda - z}{1 - \overline{\lambda} z} \Big)^n \, dz = \lambda^n .$$

Finally $n > l$ implies $L_{-n,-l} = 0$, as the integrand is a holomorphic function. □

Another view of the above is the following. For $n, l > 0$ the Cauchy formula implies that $l! \cdot L_{-n,-l}$ is the l-th derivative of $\tau(z)^n = z^n \cdot h(z)$ at $z = 0$ with $h(z) = (\lambda - z)^n/(1 - \overline{\lambda} z)^n$ a holomorphic function on $\mathbb{D}$, and therefore $L_{-n,-l}$ vanishes for $0 \le l < n$. For $l \ge n \ge 0$ one obtains $L_{-n,-l} = h^{(l-n)}(0)/(l-n)!$, which can be calculated explicitly[6]. In particular $L_{-n,-n} = h(0) = \lambda^n$ and we conclude that $L^{(N)}$ has the following upper-lower triangular matrix structure

$$L^{(N)} = \begin{pmatrix} \lambda^N & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & 0 & 0 & \vdots & \ddots & \vdots \\ * & * & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & \overline{\lambda} & * & * \\ \vdots & \ddots & \vdots & 0 & 0 & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & \overline{\lambda}^N \end{pmatrix} . \tag{3.20}$$

Clearly, the spectrum of $L^{(N)}$ is given by the diagonal elements $(L^{(N)})_{n,n}$, that is,

$$\sigma(L^{(N)}) = \{1\} \cup \{\lambda^n : n = 1, \dots, N\} \cup \{\overline{\lambda}^n : n = 1, \dots, N\}.$$

Moreover, the triangular structure of $L^{(N)}$ implies $\mathcal{L}(H_N) \subseteq H_N$.

We are now able to prove the main statement of this chapter.

Proof of Theorem 3.3.1. For every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, the map τ in (3.18) is an analytic expanding circle map. We can choose $A, A' \in \mathcal{A}$ satisfying (3.14) such that the associated $\mathcal{L}\colon H^\infty(A) \to H^\infty(A)$ admits the factorisation (3.16). As $\mathcal{L}(H_N) \subseteq H_N$ for every $N \in \mathbb{N}_0$ and by Lemma 3.2.5 $\lim_{N\to\infty} \|J - J_N\|_{H^\infty(A) \to H^\infty(A')} = 0$ with $J_N = JP_N$, Lemma 1.4.6 implies that the spectrum of $\mathcal{L}$ consists of eigenvalues, together with zero, given by

$$\sigma(\mathcal{L}) = \mathrm{cl}\big(\bigcup \sigma(\mathcal{L}|_{H_N})\big) = \mathrm{cl}\big(\bigcup \sigma(L^{(N)})\big) = \{1\} \cup \{\lambda^n : n \in \mathbb{N}\} \cup \{\overline{\lambda}^n : n \in \mathbb{N}\} \cup \{0\} .$$

The assertions concerning the multiplicities of the eigenvalues of $\mathcal{L}$ follow from the corresponding properties of $L^{(N)}$ and [**27**, XI.9.5]. □

[6]For $n > 0$ and $l \in \mathbb{Z}$ we have

$$(L^{(N)})_{n,l} = \begin{cases} (-1)^{n-l} (\overline{\lambda})^{2n-l} \sum_{m=0}^{l-n} \binom{l-m-1}{n-1} \binom{n}{m} (-|\lambda|^2)^{l-n-m} & \text{if } l \le 2n, \\ (-1)^n \lambda^{l-2n} \sum_{m=0}^{n} \binom{l-m-1}{n-1} \binom{n}{m} (-|\lambda|^2)^{n-m} & \text{if } l > 2n. \end{cases}$$

REMARK 3.3.3. To the best of the author's knowledge, these are the first examples of analytic circle maps for which the spectrum of the associated transfer operator consists of infinitely many distinct nonzero eigenvalues (on the space of holomorphic functions). Previously, the only examples of circle maps with known spectrum were those of the form $z \mapsto z^n$ for $n \geq 2$ (for $n = 2$ corresponding to $\lambda = 0$ in the above theorem). The spectra of the corresponding transfer operators when acting on analytic functions coincide with the two-point set $\{0, 1\}$ for all n. See [**7**, Ex. 2.15] for a proof when $n = 2$; the general case can be proved along the same lines. Alternatively, this will follow from Theorem 4.3.4.

3.4. Circle maps considered on an interval

In the previous section we have considered the transfer operator $\mathcal{L}_{\mathbb{T}} \colon H^\infty(A) \to H^\infty(A)$ associated to an analytic expanding circle map $\tau \colon \mathbb{T} \to \mathbb{T}$, which maps the space of bounded holomorphic functions on an appropriately chosen annulus $A \in \mathcal{A}$ compactly to itself. The circle map τ gives rise to a map T on an interval $I = [x_0, x_1]$, chosen such that a fixed point z_0 of τ corresponds to the interval endpoint x_0. Choosing a suitable complex neighbourhood D of I, we shall now study the spectral properties of $\mathcal{L}_I \colon H^\infty(D) \to H^\infty(D)$, the transfer operator corresponding to T.

More precisely, let $T \colon I \to I$ denote the interval map arising from the circle map $\tau \colon \mathbb{T} \to \mathbb{T}$ via $p \circ T = \tau \circ p$ with a projection $p \colon I \to \mathbb{T}$ satisfying[7] $p(x_0) = z_0$. Let $\{\Phi_1, \ldots, \Phi_K\}$ be the set of inverse branches of T. With slight abuse of notation we keep writing T and Φ_k for the respective analytic extensions to neighbourhoods containing I. Since τ is an analytic K-covering, we have the matching conditions (with suitable labelling of the inverse branches)

$$\begin{aligned} \Phi_1(x_0) = x_0\,, \qquad & \Phi_K(x_1) = x_1\,, \qquad \Phi_1^{(n)}(x_0) = \Phi_K^{(n)}(x_1)\,, \\ \Phi_{k+1}(x_0) = \Phi_k(x_1)\,, \quad & \Phi_{k+1}^{(n)}(x_0) = \Phi_k^{(n)}(x_1) \quad \text{for } k = 1, \ldots, K-1\,, \end{aligned} \tag{3.21}$$

where for each $n \in \mathbb{N}$, we use $\Phi_k^{(n)}$ to denote the n-th derivative of Φ_k.

Since T is expanding, all inverse branches Φ_k are contractions. We can thus choose a topological disk D containing I such that $p(D) = A$ and $\Phi_k(D) \Subset D$ for all k. Then $\mathcal{L}_I \colon H^\infty(D) \to H^\infty(D)$, given by

$$\mathcal{L}_I f = \sum_{k=1}^{K} \Phi_k' \cdot (f \circ \Phi_k)\,, \tag{3.22}$$

yields a bounded operator. Moreover, $\mathcal{L}_I$ is compact (see Proposition 1.4.5 or [**11, 59**]), its spectrum consisting of countably many eigenvalues accumulating at zero only.

REMARK 3.4.1. It is perhaps not surprising that the operators $\mathcal{L}_{\mathbb{T}}$ and $\mathcal{L}_I$ are closely related. In order to see this, we define the operator $Q_p \colon H^\infty(A) \to H^\infty(D)$ by

$$(Q_p f)(x) = p'(x) f(p(x))\,.$$

[7]A suitable choice is $p(x) = e^{2\pi i \frac{(x-x_0)}{(x_1-x_0)} + i \arg z_0}$.

Clearly $p(D) = A$ implies that Q_p is injective. However, the operator Q_p is not surjective, as the image $\mathrm{im}(Q_p) = \{f \in H^\infty(D) : f^{(n)}(x_0) = f^{(n)}(x_1)\ \forall n \in \mathbb{N}_0\}$ is not all of $H^\infty(D)$. It is easy to verify that $\mathcal{L}_I$ and $\mathcal{L}_\mathbb{T}$ are related by

$$\mathcal{L}_I Q_p = Q_p \mathcal{L}_\mathbb{T}\,,$$

and that $\sigma(\mathcal{L}_\mathbb{T}) \subseteq \sigma(\mathcal{L}_I)$, which follows from the injectivity of Q_p. On the other hand, an eigenvalue of $\mathcal{L}_I$ with an eigenfunction f is also an eigenvalue of $\mathcal{L}_\mathbb{T}$ if $f \in \mathrm{im}(Q_p)$.

The following lemma connects the spectrum of $\mathcal{L}_I$ with the spectrum of $\mathcal{L}_\mathbb{T}$. This result is mentioned in the introduction of [**43**] together with a proof based on the theory of Fredholm determinants.

Lemma 3.4.2. *Suppose that τ is an analytic expanding circle map and $T\colon I \to I$ the corresponding interval map fixing the interval endpoint x_0. Let $\mathcal{L}_\mathbb{T}$ and $\mathcal{L}_I$ be the corresponding transfer operators as defined above. Then the spectrum of $\mathcal{L}_I$ is given by*

$$\sigma(\mathcal{L}_I) = \sigma(\mathcal{L}_\mathbb{T}) \cup \left\{\, (T'(x_0))^{-n}\ :\ n \in \mathbb{N} \,\right\}\,.$$

Informally, the additional eigenvalues of $\mathcal{L}_I$ can be explained by the observation that the fixed point z_0 of τ corresponds to the two fixed points of T at the interval endpoints. The Fredholm determinants of $\mathcal{L}_I$ and $\mathcal{L}_\mathbb{T}$ can be written in terms of the traces of $\mathcal{L}_I^n$ and $\mathcal{L}_\mathbb{T}^n$, which in turn can be expressed in terms of fixed point multipliers of T^n and τ^n. The two determinants differ by a factor corresponding to the additional fixed point of T. The zeros of this factor are the reciprocals of the additional eigenvalues of $\mathcal{L}_I$. Here we shall give a short alternative proof.

Proof of Lemma 3.4.2. Let $H^\infty(D)^*$ denote the strong dual of $H^\infty(D)$, that is, the space of continuous linear functionals on $H^\infty(D)$ equipped with the topology of uniform convergence on the unit ball. Let $\mathcal{L}_I^*\colon H^\infty(D)^* \to H^\infty(D)^*$ denote the adjoint operator of $\mathcal{L}_I$, that is,

$$(\mathcal{L}_I^* l)(f) = l(\mathcal{L}_I f) \quad \text{for all } l \in H^\infty(D)^* \text{ and } f \in H^\infty(D)\,.$$

For $n \in \mathbb{N}_0$, let $l_n \in H^\infty(D)^*$ be defined by

$$l_n(f) = f^{(n)}(x_1) - f^{(n)}(x_0) \quad \text{for } f \in H^\infty(D)\,.$$

It is not difficult to see that l_0 is an eigenvector of $\mathcal{L}^*$ with eigenvalue $\Phi_1'(x_0)$ since

$$\begin{aligned}
(\mathcal{L}_I^* l_0)(f) =& (\mathcal{L}_I f)(x_1) - (\mathcal{L}_I f)(x_0)\\
=& \Phi_K'(x_1)(f\circ\Phi_K)(x_1) - \Phi_1'(x_0)(f\circ\Phi_1)(x_0)\\
&+ \sum_{k=1}^{K-1}\left(\Phi_k'(x_1)(f\circ\Phi_k)(x_1) - \Phi_{k+1}'(x_0)(f\circ\Phi_{k+1})(x_0)\right)\\
=& \Phi_1'(x_0)(f(x_1) - f(x_0))\\
=& \Phi_1'(x_0) l_0(f)\,,
\end{aligned}$$

where the penultimate equality follows from (3.21). We can proceed similarly for an arbitrary $n \in \mathbb{N}$. Observe that the n-th derivative of $\mathcal{L}_I f$ is given by

$$(\mathcal{L}_I f)^{(n)} = \sum_{k=1}^{K} \sum_{m=0}^{n-1} w_{k,m} \cdot (f^{(m)} \circ \Phi_k) + \sum_{k=1}^{K} (\Phi_k')^{n+1} \cdot (f^{(n)} \circ \Phi_k),$$

where each $w_{k,m}$ is a weight function composed of derivatives of Φ_k of order up to $n - m + 1$ satisfying $w_{k,m}(x_1) = w_{k+1,m}(x_0)$ for $k = 1, \ldots, K-1$ in analogy with (3.21). A calculation similar to the above yields

$$\begin{aligned}(\mathcal{L}_I^* l_n)(f) =& (\mathcal{L}_I f)^{(n)}(x_1) - (\mathcal{L}_I f)^{(n)}(x_0) \\ =& \sum_{m=0}^{n-1} w_{1,m}(x_0) l_m(f) + (\Phi_1'(x_0))^{n+1} l_n(f). \end{aligned} \tag{3.23}$$

It follows that $\mathcal{L}_I^* V_n \subseteq V_n$, where $V_n = \operatorname{span}\{l_0, \ldots, l_n\}$ for each n. Thus $(\Phi_1'(x_0))^n$ is an eigenvalue of $\mathcal{L}_I^*$, and hence of $\mathcal{L}_I$. As $T'(x_0) = 1/\Phi_1'(x_0)$ and every eigenvalue of $\mathcal{L}_\mathbb{T}$ is an eigenvalue of $\mathcal{L}_I$, we have shown

$$\sigma(\mathcal{L}_\mathbb{T}) \cup \{T'(x_0)^{-n} : n \in \mathbb{N}\} \subseteq \sigma(\mathcal{L}_I).$$

For the converse inclusion recall Remark 3.4.1 and assume that $f \in H^\infty(D)$ is an eigenfunction of $\mathcal{L}_I$ with eigenvalue μ and $f \notin \operatorname{im}(Q_p)$. It follows that there is $N \in \mathbb{N}_0$ such that $f^{(N)}(x_0) \neq f^{(N)}(x_1)$ and $f^{(n)}(x_0) = f^{(n)}(x_1)$ for $0 \leq n < N$, from which $l_n(f) = 0$ for $0 \leq n < N$. Since $\mathcal{L}_I f = \mu f$, this implies

$$l_N(\mu f) = l_N(\mathcal{L}_I f) = (\Phi_1'(x_0))^{N+1} l_N(f).$$

As l_N is linear and nonzero, it follows that $\mu = (\Phi_1'(x_0))^{N+1} = (T'(x_0))^{-N-1}$. □

REMARK 3.4.3. The eigenfunctions of $\mathcal{L}_I^*$ corresponding to the eigenvalues $T'(x_0)^{-n}$ with $n \in \mathbb{N}$ can be deduced from the upper triangular matrix representation of the restriction of $\mathcal{L}_I^*$ to the space $V_n = \operatorname{span}\{l_0, \ldots, l_n\}$. If T is the doubling map $T(x) = 2x \bmod 1$ on $I = [0, 1]$, then all higher derivatives of Φ_k vanish. Hence, the respective eigenfunctions of $\mathcal{L}_I^*$ are precisely l_n as $w_{1,m} = 0$ for $m = 0, \ldots, n-1$, and consequently (3.23) reduces to $(\mathcal{L}_I^* l_n)(f) = (\Phi_1'(x_0))^{n+1} l_n(f)$. Moreover, it is well known (see, for example, [**3**]) that the corresponding eigenfunctions of $\mathcal{L}_I$ are given by Bernoulli polynomials.

We can now apply this result to the interval maps introduced in Section 3.1, which we view as arising from analytic expanding circle maps. Let $I = [-1, 1]$ and $\lambda \in \mathbb{R}$ with $|\lambda| < 1$, then the interval map T arising from τ in (3.18) fixes the interval endpoint $x_0 = -1$ with $1/T'(-1) = (\lambda + 1)/2$. By Theorem 3.3.1 and Lemma 3.4.2, the eigenvalues of $\mathcal{L}_I$ can be divided into two classes, those given by the eigenvalues of $\mathcal{L}_\mathbb{T}$ (each of multiplicity 2, except the eigenvalue 1 of multiplicity 1) and those given by the powers of the inverse multiplier of the fixed point x_0, that is,

$$\sigma(\mathcal{L}_I) = \Big(\{\lambda^n : n \in \mathbb{N}_0\} \cup \{0\}\Big) \cup \Big\{\Big(\frac{\lambda+1}{2}\Big)^n : n \in \mathbb{N}\Big\}, \tag{3.24}$$

see Figure 3.4.

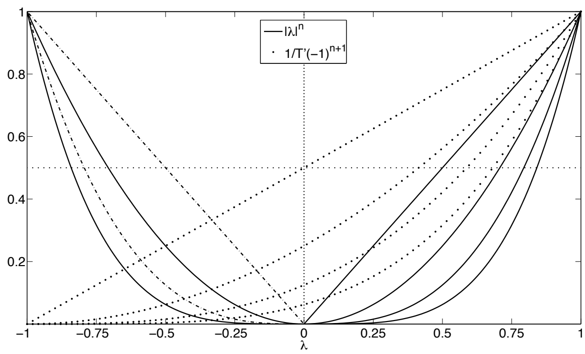

FIGURE 3.4. For each $\lambda \in (-1, 1)$ and for $n = 0, \ldots, 4$ the eigenvalues in the spectrum (3.24) of $\mathcal{L}_I$ are plotted (in modulus). These are comprised of the eigenvalues λ^n of $\mathcal{L}_{\mathbb{T}}$ (solid line for $\lambda^n > 0$, and dashed for $\lambda^n < 0$) and the eigenvalues $1/(T'(-1))^{n+1}$ of $\mathcal{L}_I$. Note that the case $\lambda = 0$ corresponds to the doubling map.

Considering $\lambda \in \mathbb{C}$, say $\lambda = |\lambda| e^{i\alpha}$ with $|\lambda| < 1$, the fixed point of τ is $z_0 = (\lambda - 1)/(1 - \overline{\lambda}) \in \mathbb{T}$ with

$$T'(-1) = \tau'(z_0) = \frac{\lambda + \overline{\lambda} - 2}{\lambda\overline{\lambda} - 1} = \frac{2(|\lambda|\cos(\alpha) - 1)}{|\lambda|^2 - 1}.$$

As above, the spectrum of $\mathcal{L}_I$ splits into two parts:

$$\sigma(\mathcal{L}_I) = \Big(\{1\} \cup \{\lambda^n : n \in \mathbb{N}\} \cup \{\overline{\lambda}^n : n \in \mathbb{N}\} \cup \{0\}\Big) \cup \left\{ \Big(\frac{|\lambda|^2 - 1}{2(|\lambda|\cos(\alpha) - 1)}\Big)^n : n \in \mathbb{N} \right\}.$$

Note that for $\lambda \notin \mathbb{R}$ the transfer operator $\mathcal{L}_I$ associated to T has countably infinitely many nonreal eigenvalues of arbitrarily small modulus. This provides counterexamples to the following conjecture.

CONJECTURE 3.4.4 (Weak variant of Mayer's conjecture in dimension one). Let $\Omega \subset \mathbb{C}$ be a bounded domain with $\Omega_{\mathbb{R}} = \Omega \cap \mathbb{R} \neq \emptyset$ and $\Phi_k \colon \Omega \to \Omega$ contracting holomorphic mappings with their unique fixed points z_k^* in $\Omega_{\mathbb{R}}$. If the $\Phi_k'(z_k^*)$ are real, then all eigenvalues of the corresponding transfer operator $\mathcal{L}_I$ in (3.22) with small enough modulus are real.

REMARK 3.4.5. Mayer [**57**] originally conjectured that transfer operators satisfying the hypotheses of the above conjecture have real spectra. Counterexamples to Mayer's conjecture were given by Levin in [**48**] which led to the above weakening of the conjecture.

Reality of spectra has been studied in a few concrete examples. For the Gauss map, Mayer showed that the eigenvalues of the transfer operator (on an appropriately defined function space) are real and tend to zero exponentially fast, see [**58**] or the survey [**59**]. Another prominent example is the linearised Feigenbaum period doubling operator, for which numerical observations [**4, 20**] suggest the spectrum to be real. Furthermore, transfer operators of expanding interval maps with one 'dominating' branch have real spectra, as shown by Rugh [**75**] using a perturbative approach.

To the best of the author's knowledge these are the first examples of nontrivial circle and interval maps for which the entire spectrum of the associated transfer operator is known explicitly. Certain conjectures were previously hard to test, but might now be more accessible.

CHAPTER 4

Spectral structure for finite Blaschke products

The family of analytic circle maps considered in the previous chapter belongs to a special class of circle maps. These are finite Blaschke products, a class of rational maps on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ given by finite products of Möbius maps preserving the unit circle and the unit disk (see Section 4.3 for definitions and properties). The beauty of Blaschke products is that they partition the Riemann sphere into simple dynamically invariant regions: the unit circle, the unit disk and the exterior disk in $\hat{\mathbb{C}}$. As it turns out, these properties guarantee the diagonal block structure representation (3.20) of $\mathcal{L}$, which was used to determine the spectrum of $\mathcal{L}$ in the previous chapter.

The purpose of this chapter is to uncover the underlying structure of transfer operators associated to analytic expanding circle maps, and to deduce the entire spectrum for those circle maps which arise from finite Blaschke products. The strategy relies on the fact that the spectrum of $\mathcal{L}$ can be understood by passing to its (Banach space) adjoint $\mathcal{L}^*$. This strategy has been explored in the context of Ruelle operators acting on the space of functions locally analytic on the Julia set of a rational function, see [**9, 49, 50, 90**]; in particular, explicit expressions for Fredholm determinants of certain Ruelle operators have been derived. In our setting of analytic expanding circle maps, we adopt a similar approach, that is, we analyse the spectrum of $\mathcal{L}$ by deriving a natural explicit representation of $\mathcal{L}^*$ (Proposition 4.2.5).

As mentioned in Remark 1.3.3, the (Banach space) adjoint of the transfer operator defined in (1.6) is known as the Koopman operator, defined on $L^\infty(X, m)$ and given by composition with the map T. In the literature, the term 'composition operator' mostly refers to compositions with analytic functions mapping a disk into itself, a setting in which operator-theoretic properties such as boundedness, compactness, and most importantly explicit spectral information are well established (good references are [**80**] or the encyclopedia on the subject [**24**]). We will demonstrate that in our particular analytic setting of finite Blaschke products, the spectrum of the adjoint operator $\mathcal{L}^*$ can be deduced by studying the spectrum of composition operators on spaces of holomorphic functions on the regions left invariant under Blaschke products.

The main result of this chapter (Theorem 4.3.4) can be summarised as follows. Let B be a finite Blaschke product such that its restriction τ to the unit circle $\mathbb{T}$ is an expanding circle map. Denote by $H^2(A)$ the Hardy-Hilbert space of functions which are holomorphic on some suitable annulus A (containing $\mathbb{T}$) and square integrable on its boundary ∂A (see Definition 4.1.1). Then the transfer operator $\mathcal{L}$ associated to τ

is compact on $H^2(A)$, with spectrum

$$\sigma(\mathcal{L}) = \{1\} \cup \{\lambda(z_0)^n : n \in \mathbb{N}\} \cup \{\overline{\lambda(z_0)}^n : n \in \mathbb{N}\} \cup \{0\},$$

where $\lambda(z_0)$ is the multiplier of the unique attracting fixed point z_0 of B in the unit disk. This implies that for finite Blaschke products which give rise to analytic expanding circle maps, the derivative of the fixed point in the unit disk completely determines the spectrum of $\mathcal{L}$.

This chapter is organised as follows. In Section 4.1.1, we review basic definitions and facts about Hardy-Hilbert spaces on annuli. The following Section 4.1.2 is devoted to analytic expanding circle maps and their transfer operators. In Section 4.2, we explicitly derive the structure of the corresponding adjoint operators after having established a suitable representation of the dual space. This structure is then used in Section 4.3 in order to obtain the spectrum of transfer operators associated to analytic expanding circle maps arising from finite Blaschke products, thus proving the main theorem of this chapter. Section 4.4 presents applications of the main theorem to expanding interval maps with arbitrarily small subleading eigenvalue, but bounded Lyapunov exponent. In this way, we return to the question raised in Chapter 2 and show that a relation between Lyapunov exponents and mixing rates of the type in Proposition 2.2.9 cannot be generalised to all nonlinear expanding maps.

The results of this chapter are contained in [**13**] and [**84**].

4.1. Transfer operators on Hardy-Hilbert spaces

Given an analytic expanding circle map, we can associate with it a transfer operator $\mathcal{L}$ given in (3.10), which was shown to be compact (Proposition 3.2.4) when restricted to certain spaces of bounded holomorphic functions on a suitable annulus around $\mathbb{T}$. In this chapter it will be more convenient to consider the restriction of $\mathcal{L}$ to Hardy-Hilbert spaces, since we can use their powerful structure as Hilbert spaces. We start by introducing these spaces on annuli and disks, which will provide a convenient setting for our analysis.

4.1.1. Hardy-Hilbert spaces. Throughout this chapter $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the one point compactification of $\mathbb{C}$. For $r > 0$ we use

$$\begin{aligned} \mathbb{T}_r &= \{z \in \mathbb{C} : |z| = r\}, \\ \mathbb{T} &= \mathbb{T}_1 \end{aligned}$$

to denote circles centred at 0, and

$$\begin{aligned} D_r &= \{z \in \mathbb{C} : |z| < r\}, \\ D_r^\infty &= \{z \in \hat{\mathbb{C}} : |z| > r\}, \\ \mathbb{D} &= D_1 \end{aligned}$$

to denote disks centred at 0 and ∞.

We will use the notation $L^2(\mathbb{T}_\rho) = L^2(\mathbb{T}_\rho, d\theta/2\pi)$, where $m = d\theta/2\pi$ is the normalised one-dimensional Lebesgue measure on $\mathbb{T}_\rho$. Finally, for U an open subset of $\hat{\mathbb{C}}$ we use $\mathrm{Hol}(U)$ for the space of holomorphic functions on U. We say f is holomorphic at ∞ if $f \circ \varsigma$ is holomorphic at 0 with $\varsigma(z) = 1/z$.

Hardy-Hilbert spaces on disks and annuli are defined as follows.

Definition 4.1.1. For $\rho > 0$ and $f : \mathbb{T}_\rho \to \mathbb{C}$ write

$$M_\rho(f) = \int_0^{2\pi} |f(\rho e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \,.$$

Then the *Hardy-Hilbert spaces* on D_r and $A_{r,R}$ are given by

$$H^2(D_r) = \left\{ f \in \mathrm{Hol}(D_r) : \sup_{\rho \nearrow r} M_\rho(f) < \infty \right\},$$

and

$$H^2(A_{r,R}) = \left\{ f \in \mathrm{Hol}(A_{r,R}) : \sup_{\rho \nearrow R} M_\rho(f) + \sup_{\rho \searrow r} M_\rho(f) < \infty \right\}.$$

The Hardy-Hilbert space on the exterior disk D_R^∞ is defined accordingly, that is $f \in H^2(D_R^\infty)$ if $f \in \mathrm{Hol}(D_R^\infty)$ (or, equivalently, $f \circ \varsigma$ holomorphic on $D_{1/R}$ with $\varsigma(z) = 1/z$) and $\sup_{\rho \searrow R} M_\rho(f) < \infty$. Finally, $H_0^2(D_R^\infty) \subset H^2(D_R^\infty)$ denotes the subspace of functions vanishing at infinity.

A comprehensive account of Hardy spaces over general domains is given in the classic text [**28**]. A crisp treatment of Hardy spaces on the unit disk can be found in [**69**, Ch. 17]), while a good reference for Hardy spaces on annuli is [**79**]. We shall now collect a number of results which will be useful in what follows.

Any function in $H^2(U)$, where U is a disk or an annulus, can be extended to the boundary in the following sense. For $f \in H^2(D_r)$ there is an $f^* \in L^2(\mathbb{T}_r)$ such that

$$\lim_{\rho \nearrow r} f(\rho e^{i\theta}) = f^*(re^{i\theta}) \quad \text{for a.e. } \theta,$$

and analogously for $f \in H^2(D_R^\infty)$. Similarly, for $f \in H^2(A_{r,R})$ there are $f_1^* \in L^2(\mathbb{T}_r)$ and $f_2^* \in L^2(\mathbb{T}_R)$, with $\lim_{\rho \searrow r} f(\rho e^{i\theta}) = f_1^*(re^{i\theta})$ and $\lim_{\rho \nearrow R} f(\rho e^{i\theta}) = f_2^*(Re^{i\theta})$ for a.e. θ. It turns out that the spaces $H^2(A_{r,R})$, $H^2(D_r)$ and $H^2(D_R^\infty)$ are Hilbert spaces. The next theorem summarises the properties of $H^2(A_{r,R})$, with the cases of $H^2(D_r)$ and $H^2(D_R^\infty)$ being similar.

Theorem 4.1.2. *(a) The space $H^2(A_{r,R})$ is a Hilbert space with inner product*

$$(f, g)_{H^2(A_{r,R})} = \int_0^{2\pi} f_1^*(re^{i\theta})\overline{g_1^*(re^{i\theta})} \, \frac{d\theta}{2\pi} + \int_0^{2\pi} f_2^*(Re^{i\theta})\overline{g_2^*(Re^{i\theta})} \, \frac{d\theta}{2\pi}.$$

(b) An orthonormal basis for $H^2(A_{r,R})$ is given by $\mathcal{E} = \{e_n : n \in \mathbb{Z}\}$, where

$$e_n(z) = \frac{z^n}{d_n} \quad \textit{with } d_n = \sqrt{r^{2n} + R^{2n}}. \tag{4.1}$$

For $f \in \mathrm{Hol}(A_{r,R})$ it follows that $f \in H^2(A_{r,R})$ if and only if

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e_n(z) \quad \textit{with} \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty,$$

where the coefficients are given by $c_n = c_n(f) = (f, e_n)_{H^2(A_{r,R})}$.

(c) *Every $f \in H^2(A_{r,R})$ satisfies $\|f\|^2_{H^2(A_{r,R})} = \|f_1^*\|^2_{L^2(\mathbb{T}_r)} + \|f_2^*\|^2_{L^2(\mathbb{T}_R)}$, and*

$$\|f\|^2_{H^2(A_{r,R})} = (f, f)_{H^2(A_{r,R})} = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Similar statements hold for $H^2(D_r)$, with inner product given by

$$(f, g)_{H^2(D_r)} = \int_0^{2\pi} f^*(re^{i\theta})\overline{g^*(re^{i\theta})}\, \frac{d\theta}{2\pi},$$

and analogously for $H^2(D_R^\infty)$.

NOTATION 4.1.3. In order to avoid cumbersome notation, we shall write $f(z)$ instead of $f^*(z)$ for z on the boundary of the domain.

4.1.2. Factorisation of transfer operators. We can now define transfer operators associated to analytic expanding circle maps on suitable Hardy-Hilbert spaces. Let τ be an analytic expanding circle map. The expansivity of τ and Lemma 3.2.2 allow us to choose A_0, A' and A in $\mathcal{A}$ with

$$A_0 \Subset A' \Subset A \text{ and } \tau(\partial A_0) \cap \mathrm{cl}(A) = \emptyset. \tag{4.2}$$

As we shall see presently, by adapting the factorisation argument explained in Section 1.4 and applied in Section 3.2, the above choices of the annuli guarantee that the associated transfer operator $\mathcal{L}$ in (3.10) is a well-defined linear operator which maps $H^2(A)$ compactly to itself. We can write $\mathcal{L} = \tilde{\mathcal{L}}\mathcal{J}$, where $\tilde{\mathcal{L}} \colon H^\infty(A') \to H^2(A)$ is a lifted transfer operator given by the same functional expression (3.10) and $\mathcal{J} \colon H^2(A) \to H^\infty(A')$ is the canonical embedding:

$$\begin{tikzcd}
H^\infty(A') \arrow[dr, "\tilde{\mathcal{L}}"] & \\
H^2(A) \arrow[u, hook, "\mathcal{J}"] \arrow[r, "\mathcal{L}"'] & H^2(A)
\end{tikzcd}$$

Note that compared to the diagram (1.19), we use $H^\infty(A')$ instead of $H^2(A')$ as this choice allows for an easy proof of continuity of $\tilde{\mathcal{L}}$ in Lemma 4.1.4.

Let R, R' denote the radii of the circles forming the 'exterior' boundaries, and r, r' the radii of the circles forming the 'interior' boundaries of A and A', respectively, that is, $A = A_{r,R}$ and $A' = A_{r',R'}$.

LEMMA 4.1.4. *The transfer operator $\tilde{\mathcal{L}}$ given by* (3.10) *maps $H^\infty(A')$ continuously to $H^2(A)$.*

PROOF. We can factorise $\tilde{\mathcal{L}}$ as $\tilde{\mathcal{L}} = \hat{\mathcal{J}}\hat{\mathcal{L}}$, where $\hat{\mathcal{L}}\colon H^\infty(A') \to H^\infty(A)$, given by the functional expression (3.10), is continuous by Lemma 3.2.3, and $\hat{\mathcal{J}}\colon H^\infty(A) \hookrightarrow H^2(A)$ is the canonical embedding. $\square$

Next, we establish compactness of $\mathcal{J}\colon H^2(A) \hookrightarrow H^\infty(A')$ given by

$$(\mathcal{J}f)(z) = f(z) \quad \text{for } z \in A'.$$

Let $\{e_n : n \in \mathbb{Z}\}$ be the orthonormal basis for $H^2(A)$ given by (4.1), then any $f \in H^2(A)$ can be uniquely expressed as $f = \sum_{n\in\mathbb{Z}} c_n(f)e_n$. For $N \in \mathbb{N}$ define the finite rank operator $\mathcal{J}_N\colon H^2(A) \to H^\infty(A')$ by

$$(\mathcal{J}_N f)(z) = \sum_{n=-N+1}^{N-1} c_n(f)e_n(z) \quad \text{for } z \in A'.$$

LEMMA 4.1.5. *Let $\mathcal{J}$ and $\mathcal{J}_N$ be as above. Then*

$$\lim_{N\to\infty} \|\mathcal{J} - \mathcal{J}_N\|_{H^2(A)\to H^\infty(A')} = 0.$$

In particular, the embedding $\mathcal{J}$ is compact.

PROOF. For $z \in A'$, it follows by the Cauchy-Schwarz inequality that

$$\begin{aligned}
|(\mathcal{J}f)(z) - (\mathcal{J}_N f)(z)| &\le \left(\sum_{|n|\ge N} |c_n(f)|^2\right)^{1/2} \left(\sum_{|n|\ge N} |e_n(z)|^2\right)^{1/2} \\
&\le \|f\|_{H^2(A)} \left(\sum_{|n|\ge N} \frac{|z^n|^2}{r^{2n} + R^{2n}}\right)^{1/2} \\
&\le \|f\|_{H^2(A)} \left(\sum_{n\ge N} \left|\frac{z}{R}\right|^{2n} + \sum_{n\ge N} \left|\frac{r}{z}\right|^{2n}\right)^{1/2}.
\end{aligned}$$

Thus

$$\|\mathcal{J}f - \mathcal{J}_N f\|_{H^\infty(A')} \le \|f\|_{H^2(A)} \left(\left(\frac{R'}{R}\right)^{2N} \frac{1}{1-(\frac{R'}{R})^2} + \left(\frac{r}{r'}\right)^{2N} \frac{1}{1-(\frac{r}{r'})^2}\right)^{1/2},$$

and the assertions follow. $\square$

The factorisation $\mathcal{L} = \tilde{\mathcal{L}}\mathcal{J}$ together with Lemmas 4.1.4 and 4.1.5 now imply the following result.

PROPOSITION 4.1.6. *The transfer operator $\mathcal{L}\colon H^2(A) \to H^2(A)$ given by the functional expression* (3.10) *is compact.*

4.2. Adjoint operator

A central step in showing our main result is to find an appropriate representation of the dual space on which the adjoint of the transfer operator has a simple structure.

For the remainder of this section we set $A = A_{r,R}$ and denote by $H^2(A)^*$ the strong dual of $H^2(A)$, that is, the space of continuous linear functionals on $H^2(A)$ equipped with the topology of uniform convergence on the unit ball. We will show that $H^2(A)^*$ is isomorphic to the topological direct sum $H^2(D_r) \oplus H^2_0(D^\infty_R)$, equipped with the norm $\|(h_1, h_2)\|^2 = \|h_1\|^2_{H^2(D_r)} + \|h_2\|^2_{H^2_0(D^\infty_R)}$. Similar representations of the duals of Hardy spaces for multiply connected regions can be found in [**68**, Prop. 3]. The present setup is sufficiently simple to allow for a short proof of the representation.

PROPOSITION 4.2.1. *The dual space $H^2(A)^*$ is isomorphic to $H^2(D_r) \oplus H^2_0(D^\infty_R)$ with the isomorphism given by*

$$J\colon H^2(D_r) \oplus H^2_0(D^\infty_R) \to H^2(A)^*$$
$$(h_1, h_2) \mapsto l,$$

where

$$l(f) = \frac{1}{2\pi i}\int_{\mathbb{T}_r} f(z)h_1(z)\,dz + \frac{1}{2\pi i}\int_{\mathbb{T}_R} f(z)h_2(z)\,dz \quad (f \in H^2(A)). \tag{4.3}$$

PROOF. We will first show that (4.3) defines a continuous functional $l \in H^2(A)^*$ and that J is a bounded linear operator. In order to see this note that for any $(h_1, h_2) \in H^2(D_r) \oplus H^2_0(D^\infty_R)$ the linear functional $l = J(h_1, h_2)$ is bounded, since for any $f \in H^2(A)$ with $\|f\|_{H^2(A)} \leq 1$

$$|l(f)| \leq \left(r\,\|h_1\|_{H^2(D_r)} + R\,\|h_2\|_{H^2_0(D^\infty_R)}\right).$$

It follows that

$$\|J(h_1, h_2)\|_{H^2(A)^*} \leq \sqrt{r^2 + R^2}\sqrt{\|h_1\|^2_{H^2(D_r)} + \|h_2\|^2_{H^2_0(D^\infty_R)}}$$

and $\|J\|_{H^2(D_r)\oplus H^2_0(D^\infty_R)\to H^2(A)^*} \leq \sqrt{r^2 + R^2}$. Hence, J is well defined and bounded.

For injectivity, we suppose that $l = J(h_1, h_2) = 0$ and show that $h_1 = 0$ and $h_2 = 0$. In order to see this note that any $(h_1, h_2) \in H^2(D_r) \oplus H^2_0(D^\infty_R)$ can be written as $h_1(z) = \sum_{n=0}^\infty a_n z^n$ and $h_2(z) = \sum_{n=1}^\infty a_{-n} z^{-n}$ with suitable coefficients $a_n \in \mathbb{C}$. Now let

$$\mathcal{E} = \{e_n : n \in \mathbb{Z}\} \quad \text{with} \quad e_n(z) = \frac{z^n}{d_n}$$

denote the orthonormal basis of $H^2(A)$ given in (4.1). A short calculation using Lebesgue dominated convergence shows that

$$0 = (J(h_1, h_2))(e_n) = \frac{a_{-n-1}}{d_n} \quad \text{for all } n \in \mathbb{Z}, \tag{4.4}$$

which implies $h_1 = 0$ and $h_2 = 0$. Thus J is injective.

Finally, in order to show that J is surjective, fix $l \in H^2(A)^*$. We will construct $(h_1, h_2) \in H^2(D_r) \oplus H^2_0(D^\infty_R)$ such that $J(h_1, h_2) = l$.

By the Riesz representation theorem there is a unique $g \in H^2(A)$ such that $l(f) = (f, g)_{H^2(A)}$ for all $f \in H^2(A)$. Moreover, g can be uniquely expressed as

$g = \sum_{n\in\mathbb{Z}} c_n(g)e_n$. Now define

$$
\begin{aligned}
h_1(z) &= \sum_{n=0}^{\infty} \overline{c_{-n-1}(g)} d_{-n-1} z^n \quad \text{for } z \in D_r, \\
h_2(z) &= \sum_{n=1}^{\infty} \overline{c_{n-1}(g)} d_{n-1} z^{-n} \quad \text{for } z \in D_R^\infty.
\end{aligned}
\tag{4.5}
$$

Using $\|g\|^2_{H^2(A)} = \sum_{n\in\mathbb{Z}} |c_n(g)|^2 < \infty$, it follows that $h_1 \in H^2(D_r)$ and $h_2 \in H^2_0(D_R^\infty)$.

Combining (4.4) and (4.5) we obtain

$$(J(h_1,h_2))(e_n) = \frac{a_{-n-1}}{d_n} = \frac{\overline{c_n(g)d_n}}{d_n} = \overline{c_n(g)} = (e_n, g)_{H^2(A)}$$

for every $n \in \mathbb{Z}$. Since the above equality also holds for all finite linear combinations of elements in $\mathcal{E}$, the continuity of J implies

$$(J(h_1,h_2))(f) = (f,g)_{H^2(A)} = l(f)$$

for all $f \in H^2(A)$. Thus J is surjective. $\square$

Remark 4.2.2. The inverse J^{-1} of J can be obtained using the kernel $K_z \in H^2(A)$ defined by $K_z(w) = 1/(z-w)$ for $z \in \hat{\mathbb{C}} \setminus \mathrm{cl}(A)$. More precisely, J^{-1} is given by $l \mapsto (h_1, h_2)$, where $h_1(z) = l(-K_z)$ for $z \in D_r$ and $h_2(z) = l(K_z)$ for $z \in D_R^\infty$, see also [**68**, p. 159]. The inverse J^{-1} is bounded by the bounded inverse theorem, and its norm can be estimated directly by using (4.5):

$$\left\|J^{-1}l\right\|^2_{H^2(D_r)\oplus H^2_0(D_R^\infty)} = \|h_1\|^2_{H^2(D_r)} + \|h_2\|^2_{H^2_0(D_R^\infty)} \le \left(\frac{1}{r^2} + \frac{1}{R^2}\right) \|l\|^2_{H^2(A)^*}.$$

Returning to the setting of Section 4.1.2, let τ be an analytic expanding circle map and $A = A_{r,R} \in \mathcal{A}$ an annulus satisfying (4.2) such that the associated transfer operator $\mathcal{L}\colon H^2(A) \to H^2(A)$ is well defined and compact. We shall first assume that τ is orientation-preserving and comment on the orientation-reversing case at the end of this section. Using the representation of the dual space $H^2(A)^*$ obtained in the previous lemma, we shall shortly derive an explicit form for the adjoint operator of $\mathcal{L}$.

Before doing so we require some more notation. Define $C^{(r)}\colon H^2(D_r) \to L^2(\mathbb{T}_r)$ by

$$(C^{(r)}h)(z) = h(\tau(z)) \quad \text{for } z \in \mathbb{T}_r\,, \tag{4.6}$$

and $C^{(R)}\colon H^2_0(D_R^\infty) \to L^2(\mathbb{T}_R)$ by

$$(C^{(R)}h)(z) = h(\tau(z)) \quad \text{for } z \in \mathbb{T}_R\,. \tag{4.7}$$

These operators could be called 'composition operators', but we restrict the use of this term to the case of operators mapping the space of holomorphic functions to itself, see Section 4.3. It turns out that $C^{(r)}$ and $C^{(R)}$ are compact, the proof of which relies on the following fact.

LEMMA 4.2.3. *Let K be a compact subset of a disk D in $\mathbb{C}$. Then there exists a constant c_K depending on K only such that for any $f \in H^2(D)$*

$$\sup_{z\in K} |f(z)| \leq c_K \|f\|_{H^2(D)} .$$

PROOF. This follows, for example, from [**12**, Lem. 2.9], or by a calculation using the Cauchy-Schwarz inequality, similar to the proof of Lemma 4.1.5.

Let R be the radius of the disk D, then there exists $r < R$ such that $K \subseteq D_r$. Denote by $\{e_n : n \in \mathbb{N}\}$ with $e_n(z) = z^n/R^n$ an orthonormal basis of $H^2(D)$. Then for any $z \in K$ we have

$$|f(z)| \leq \sum_{n=0}^{\infty} |c_n(f)e_n(z)| \leq \left(\sum_{n=0}^{\infty} |c_n(f)|^2\right)^{1/2} \left(\sum_{n=0}^{\infty} |e_n(z)|^2\right)^{1/2} \leq \frac{\|f\|_{H^2(D)}}{1-(\frac{r}{R})^2} ,$$

which finishes the proof with $c_K = \frac{1}{1-(r/R)^2}$. □

We now have the following.

LEMMA 4.2.4. *The operators $C^{(r)}$ and $C^{(R)}$ are compact.*

PROOF. The choice of $A = A_{r,R}$ in (4.2) implies that $r_0 = \sup_{z\in\mathbb{T}_r} |\tau(z)| < r$, and we can choose a disk $D_{r'}$ with $D_{r_0} \Subset D_{r'} \Subset D_r$.

Let $\tilde{C}^{(r)} : H^2(D_{r'}) \to L^2(\mathbb{T}_r)$ be defined by the functional expression as in (4.6), but now considered on $H^2(D_{r'})$. The operator is continuous since

$$\left\|\tilde{C}^{(r)}h\right\|_{L^2(\mathbb{T}_r)} \leq \sup_{z\in\tau(\mathbb{T}_r)} |h(z)| \leq \sup_{z\in\mathrm{cl}(D_{r_0})} |h(z)| \leq c_K \|h\|_{H^2(D_{r'})} ,$$

where we have used Lemma 4.2.3 with $K = \mathrm{cl}(D_{r_0})$. The lemma follows since we can write $C^{(r)} = \tilde{C}^{(r)}\tilde{\mathcal{J}}$ with $\tilde{\mathcal{J}} : H^2(D_r) \hookrightarrow H^2(D_{r'})$ denoting the canonical embedding, which is compact (see, for example, [**12**, Lem. 2.9]). The argument for $C^{(R)}$ is similar. □

Next, we need to define certain projection operators on $L^2(\mathbb{T}_\rho)$. For any $g \in L^2(\mathbb{T}_\rho)$ we can write $g(z) = \sum_{n\in\mathbb{Z}} g_n z^n$, so that $g = g_+ + g_-$ with $g_+(z) = \sum_{n=0}^{\infty} g_n z^n$ and $g_-(z) = \sum_{n=1}^{\infty} g_{-n} z^{-n}$. Since $\|g\|^2_{L^2(\mathbb{T}_\rho)} = \sum_{n=-\infty}^{\infty} |g_n|^2 \rho^{2n} < \infty$, the functions g_+ and g_- can be viewed as functions in $H^2(D_\rho)$ and $H^2_0(D^\infty_\rho)$, respectively. Then we define the bounded projection operators $\Pi^{(\rho)}_+ : L^2(\mathbb{T}_\rho) \to H^2(D_\rho)$ and $\Pi^{(\rho)}_- : L^2(\mathbb{T}_\rho) \to H^2_0(D^\infty_\rho)$ by

$$\Pi^{(\rho)}_+(g) = g_+ \quad \text{and} \quad \Pi^{(\rho)}_-(g) = g_- . \tag{4.8}$$

Finally, let $\mathcal{L}^* : H^2(A)^* \to H^2(A)^*$ denote the adjoint operator of $\mathcal{L}$ in the Banach space sense, that is, $(\mathcal{L}^*l)(f) = l(\mathcal{L}f)$ for all $l \in H^2(A)^*$ and $f \in H^2(A)$. The following proposition provides an explicit representation $\mathcal{L}'$ of $\mathcal{L}^*$ via

$$\mathcal{L}' = J^{-1}\mathcal{L}^*J ,$$

as an operator on $H^2(D_r) \oplus H^2_0(D^\infty_R)$ given by compositions of $C^{(\rho)}$, $\Pi^{(\rho)}_-$ and $\Pi^{(\rho)}_+$ for $\rho = r, R$.

PROPOSITION 4.2.5. *Let $\mathcal{L}\colon H^2(A) \to H^2(A)$ be the transfer operator associated to an analytic orientation-preserving expanding circle map τ, with $A \in \mathcal{A}$ as in (4.2). Then the isomorphism J conjugates the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ to*

$$\mathcal{L}'\colon H^2(D_r) \oplus H_0^2(D_R^\infty) \to H^2(D_r) \oplus H_0^2(D_R^\infty)$$

given[1] *by*

$$\mathcal{L}' = \begin{pmatrix} \Pi_+^{(r)} C^{(r)} & \Pi_+^{(R)} C^{(R)} \\ \Pi_-^{(r)} C^{(r)} & \Pi_-^{(R)} C^{(R)} \end{pmatrix}, \tag{4.9}$$

that is $\mathcal{L}' = J^{-1}\mathcal{L}^* J$.

PROOF. We want to show that $\mathcal{L}^* J = J\mathcal{L}'$, that is,

$$(\mathcal{L}^* J(h_1, h_2))(f) = (J\mathcal{L}'(h_1, h_2))(f) \tag{4.10}$$

for all $(h_1, h_2) \in H^2(D_r) \oplus H_0^2(D_R^\infty)$ and $f \in H^2(A)$. For any such (h_1, h_2) and f, the adjoint property yields

$$\begin{aligned}(\mathcal{L}^* J(h_1, h_2))(f) &= (J(h_1, h_2))(\mathcal{L}f) \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}_r} (\mathcal{L}f)(z) h_1(z)\, dz + \frac{1}{2\pi i}\int_{\mathbb{T}_R} (\mathcal{L}f)(z) h_2(z)\, dz\ .\end{aligned}$$

Next we shall use the integral definition of $\mathcal{L}$ on $\mathbb{T}$, see (3.11), and express the above integrands in terms of compositions of h_1 and h_2 with τ. This is first done for monomials forming a basis of $H^2(A)$ and $H^2(D_r) \oplus H_0^2(D_R^\infty)$. These are holomorphic on $\mathrm{cl}(A)$ which allows us to deform the contours of integration to $\mathbb{T}$.

More precisely, let a basis for $H^2(D_r) \oplus H_0^2(D_R^\infty)$ be given by $\mathcal{P} = \{(p_n, 0) : n \in \mathbb{N}_0\} \cup \{(0, p_{-n}) : n \in \mathbb{N}\}$ with $p_n(z) = z^n$, where $p_n \in H^2(D_r)$ if $n \geq 0$ and $p_n \in H_0^2(D_R^\infty)$ if $n < 0$. Take $f \in \mathcal{E}$, where $\mathcal{E}$ is the basis for $H^2(A)$ given by (4.1). For $n \in \mathbb{N}_0$ and $(h_1, h_2) = (p_n, 0) \in \mathcal{P}$ we get

$$\begin{aligned}(\mathcal{L}^* J(h_1, 0))(f) &= \frac{1}{2\pi i}\int_{\mathbb{T}_r} (\mathcal{L}f)(z) h_1(z)\, dz \\ &\overset{(a)}{=} \frac{1}{2\pi i}\int_{\mathbb{T}} (\mathcal{L}f)(z) h_1(z)\, dz \\ &\overset{(b)}{=} \frac{1}{2\pi i}\int_{\mathbb{T}} f(z)(h_1 \circ \tau)(z)\, dz \\ &\overset{(c)}{=} \frac{1}{2\pi i}\int_{\mathbb{T}_r} f(z)(h_1 \circ \tau)(z)\, dz\, .\end{aligned}$$

[1]Note that $H^2(D_R)$ can be viewed as a subspace of $H^2(D_r)$, and similarly $H_0^2(D_r^\infty)$ as a subspace of $H_0^2(D_R^\infty)$. Thus the off-diagonal elements of $\mathcal{L}'$ are well defined.

Here, equalities (a) and (c) follow since the integrands are analytic on A and equality (b) follows by definition of $\mathcal{L}$ (see also (3.11)). Then, by the definition of $\Pi_+^{(r)}$ and $\Pi_-^{(r)}$,

$$\begin{aligned}
&(\mathcal{L}^* J(h_1,0))(f)\\
=&\frac{1}{2\pi i}\int_{\mathbb{T}_r} f(z)(\Pi_+^{(r)}(h_1\circ\tau))(z)\,dz + \frac{1}{2\pi i}\int_{\mathbb{T}_r} f(z)(\Pi_-^{(r)}(h_1\circ\tau))(z)\,dz\\
=&\frac{1}{2\pi i}\int_{\mathbb{T}_r} f(z)(\Pi_+^{(r)}(h_1\circ\tau))(z)\,dz + \frac{1}{2\pi i}\int_{\mathbb{T}_R} f(z)(\Pi_-^{(r)}(h_1\circ\tau))(z)\,dz\\
=&(J\mathcal{L}'(h_1,0))(f)\,.
\end{aligned}$$

The penultimate equality follows from the fact that $\Pi_-^{(r)}(h_1\circ\tau)\in H_0^2(D_r^\infty)$.

Analogously, for $n\in\mathbb{N}$ and $(h_1,h_2)=(0,p_{-n})\in\mathcal{P}$, the same argument shows

$$(\mathcal{L}^* J(0,h_2))(f) = (J\mathcal{L}'(0,h_2))(f)\,.$$

Hence, for $f\in\mathcal{E}$, by linearity (4.10) holds for all finite linear combinations of basis elements (h_1,h_2) in $\mathcal{P}$. Since these form a dense subspace of $H^2(D_r)\oplus H_0^2(D_R^\infty)$, and $\mathcal{L}^*$, $\mathcal{L}'$ and J are continuous operators, equality (4.10) holds for all $(h_1,h_2)\in H^2(D_r)\oplus H_0^2(D_R^\infty)$ and $f\in\mathcal{E}$. By continuity, this extends to all $f\in H^2(A)$, which completes the proof. $\square$

Remark 4.2.6. Lemma 4.2.4 and continuity of the projection operators in (4.8) imply that $\mathcal{L}'$ is compact. Note, however, that this also follows from compactness of $\mathcal{L}$ guaranteed by the choice of A in (4.2).

Remark 4.2.7. The above proposition requires only minor modifications if τ is assumed to be orientation-reversing. The operators $C^{(r)}$ and $C^{(R)}$ in (4.6) and (4.7) are replaced with $\hat{C}^{(r)}\colon H^2(D_r)\to L^2(\mathbb{T}_R)$ and $\hat{C}^{(R)}\colon H_0^2(D_R^\infty)\to L^2(\mathbb{T}_r)$, defined by

$$(\hat{C}^{(r)}h)(z) = h(\tau(z))\quad\text{for } z\in\mathbb{T}_R$$

and

$$(\hat{C}^{(R)}h)(z) = h(\tau(z))\quad\text{for } z\in\mathbb{T}_r\,,$$

which are compact by the same argument as in Lemma 4.2.4. The adjoint operator in the proposition is then represented by

$$\mathcal{L}' = \begin{pmatrix} \Pi_+^{(R)}\hat{C}^{(r)} & \Pi_+^{(r)}\hat{C}^{(R)}\\ \Pi_-^{(R)}\hat{C}^{(r)} & \Pi_-^{(r)}\hat{C}^{(R)} \end{pmatrix}.$$

The explicit representation of the adjoint operator in the above proposition facilitates the study of spectral properties for analytic expanding circle maps, and even allows us to determine the entire correlation spectrum explicitly for finite Blaschke products.

4.3. Spectrum for Blaschke products

Having discussed transfer operators $\mathcal{L}$ associated with analytic expanding circle maps and a convenient representation of the corresponding adjoint operators (Proposition 4.2.5), we shall now use this representation to obtain the full spectrum of $\mathcal{L}$ for finite Blaschke products, a class of circle maps defined as follows (see, for example, [**60**, p. 5-3]).

Definition 4.3.1. For $n \geq 2$, let $\{a_1, \ldots, a_n\}$ be a finite set of complex numbers in the open unit disk $\mathbb{D}$. A *finite Blaschke product* is a map of the form

$$B(z) = C \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i} z}\,, \tag{4.11}$$

where $|C| = 1$.

It follows from the definition that

(i) B is a meromorphic function on $\hat{\mathbb{C}}$ with zeros a_i and poles $1/\overline{a}_i$;
(ii) B is holomorphic on a neighbourhood of $\overline{\mathbb{D}}$ with $B(\mathbb{D}) = \mathbb{D}$ and $B(\mathbb{T}) = \mathbb{T}$.

Note also that a function f is holomorphic on an open neighbourhood of $\overline{\mathbb{D}}$ with $f(\mathbb{T}) = \mathbb{T}$ if and only if f is a finite Blaschke product (see, for example, [**19**, Ex. 6.12]).

Let $\tau\colon \mathbb{T} \to \mathbb{T}$ denote the restriction of a finite Blaschke product B to $\mathbb{T}$. A short calculation shows that τ is expanding if $\sum_{i=1}^{n}(1-|a_i|)/(1+|a_i|) > 1$ (see [**55**, Corollary to Prop. 1] for details). Expansiveness of τ is related to the nature of the fixed points of B, as the following result shows.

Proposition 4.3.2. *Let B and τ be as above and $|\tau'(z)| > 1$ for all $z \in \mathbb{T}$. Then B has exactly $n-1$ fixed points on $\mathbb{T}$, which are repelling, and two fixed points $z_0 \in \mathbb{D}$ and $\hat{z}_0 = 1/\overline{z}_0 \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, which are attracting.*

Moreover, there exists a unique τ-invariant probability measure μ on $\mathbb{T}$, absolutely continuous with respect to m, with the density $\varrho(z) = (1 - |z_0|^2)/|z - z_0|^2$ for $z \in \mathbb{T}$.

Proof. Using expansivity of τ, Brouwer's fixed point theorem and the Schwarz lemma imply the existence of an attracting fixed point $z_0 \in \mathbb{D}$. Since $B(z)\overline{B(1/\overline{z})} = 1$, another attracting fixed point is given by $\hat{z}_0 = 1/\overline{z_0} \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. As τ is an expanding n-covering, there are exactly $n-1$ (repelling) fixed points on $\mathbb{T}$. For details see [**65**, Prop. 2.1] and [**88**]. The second claim follows from [**55**, Thm. 1], with the explicit form of the invariant density given by a Poisson kernel, obtained using analyticity of τ on $\mathbb{D}$ and the uniqueness theorem of harmonic functions applied to Poisson integrals. $\square$

Crucial for the proof of our main theorem is the notion of a composition operator, which we briefly recall.

Definition 4.3.3. Let U be an open region in $\hat{\mathbb{C}}$. If $\psi\colon U \to U$ is holomorphic, then $C_\psi\colon \operatorname{Hol}(U) \to \operatorname{Hol}(U)$ defined by $C_\psi f = f \circ \psi$ is called a *composition operator (with symbol ψ)*.

Note that in the literature the term 'composition operator' is mostly used in the context of holomorphic functions. The operators in (4.6) do not formally fall into this category, but will turn out to be composition operators for symbols which are finite Blaschke products.

We are now able to state our main result.

THEOREM 4.3.4. *Let B be a finite Blaschke product such that $\tau = B|_{\mathbb{T}}$ is an analytic expanding circle map. Then*

(a) the transfer operator $\mathcal{L}\colon H^2(A) \to H^2(A)$ associated with τ is well defined and compact for some annulus $A \in \mathcal{A}$, and

(b) the spectrum of $\mathcal{L}\colon H^2(A) \to H^2(A)$ is given by

$$\sigma(\mathcal{L}) = \{1\} \cup \{\lambda(z_0)^n : n \in \mathbb{N}\} \cup \{\lambda(\hat{z}_0)^n : n \in \mathbb{N}\} \cup \{0\}\,, \tag{4.12}$$

where $\lambda(z_0)$ and $\lambda(\hat{z}_0) = \overline{\lambda(z_0)}$ are the multipliers[2] of the unique fixed points z_0 and $\hat{z}_0$ of B in $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, respectively.

Moreover, the algebraic multiplicity of the leading eigenvalue is 1*, while for each other nonzero eigenvalue the algebraic (and geometric) multiplicity is equal to the number of its occurrences in the list* (4.12).

PROOF. The first assertion is obvious, as τ is an analytic expanding circle map and we can choose $A = A_{r,R} \in \mathcal{A}$ as in (4.2) such that $\mathcal{L}$ is well defined and compact by the results in Section 4.1.2.

For the second claim, we will use the fact that the spectrum of $\mathcal{L}$ coincides with that of its adjoint $\mathcal{L}^*$, which together with the structure of the representation $\mathcal{L}'$ of $\mathcal{L}^*$ will allow us to deduce (4.12).

We start by observing that for the chosen A we have $B(\partial A) \cap \mathrm{cl}(A) = \emptyset$, as well as $B(D_r) \Subset D_r$ and $B(D_R^\infty) \Subset D_R^\infty$. It follows that $f \circ B \in H^2(D_r)$ for any $f \in H^2(D_r)$, and $f \circ B \in H^2(D_R^\infty)$ for any $f \in H^2(D_R^\infty)$, so that $C_B^{(r)} f = f \circ B$ and $C_B^{(R)} f = f \circ B$ define composition operators on $H^2(D_r)$ and $H^2(D_R^\infty)$, respectively. It is a standard fact that $B(D_r) \Subset D_r$ guarantees compactness of $C_B^{(r)}$ (see, for example, [**24**, pp. 128-129]), and similarly for $C_B^{(R)}$. It is also well known (see [**59**, Lem. 7.10] or [**24**, Thm. 7.20]) that all eigenvalues of a compact composition operator C_ψ are simple and are given by the nonnegative integer powers of the multiplier of the unique attracting fixed point of ψ. Hence,

$$\sigma(C_B^{(r)}) = \{\lambda(z_0)^n : n \in \mathbb{N}_0\} \cup \{0\}$$

and

$$\sigma(C_B^{(R)}) = \{\lambda(\hat{z}_0)^n : n \in \mathbb{N}_0\} \cup \{0\}\,,$$

where z_0 and $\hat{z}_0$ are the unique attracting fixed points of B in D_r and D_R^∞, respectively (see Proposition 4.3.2).

[2]Recall that the multiplier $\lambda(z^*)$ of a fixed point z^* of a rational map R is given by $R'(z^*)$ if $z^* \in \mathbb{C}$ and $1/R'(z^*)$ if $z^* = \infty$, see [**14**, p. 41]. For Blaschke products the equality $\lambda(\hat{z}_0) = \overline{\lambda(z_0)}$ follows from a straightforward calculation.

We now explain how to use these observations to determine the spectrum of $\mathcal{L}'$ given in (4.9). The projection $\Pi_+^{(r)}$ onto $H^2(D_r)$ given in (4.8) acts as the identity $I^{(r)}$ on $H^2(D_r)$ because any $f \in H^2(D_r)$ can be written as $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $C_B^{(r)}(H^2(D_r)) \subseteq H^2(D_r)$ we have $\Pi_+^{(r)} C_B^{(r)} = C_B^{(r)}$, and consequently $\Pi_-^{(r)} C_B^{(r)} = (I^{(r)} - \Pi_+^{(r)}) C_B^{(r)} = 0$. Thus the operator $\mathcal{L}'$ leaves $H^2(D_r) \oplus \{0\}$ invariant and is given by

$$\mathcal{L}' = \begin{pmatrix} C_B^{(r)} & \Pi_+^{(R)} C_B^{(R)} \\ 0 & \Pi_-^{(R)} C_B^{(R)} \end{pmatrix}. \tag{4.13}$$

Further, any f in $H_0^2(D_R^\infty)$ vanishes at ∞, but not necessarily $C_B^{(R)} f$. In particular, $(C_B^{(R)} f)(\infty) = f(B(\infty)) = 0$ for all $f \in H_0^2(D_R^\infty)$ only if $B(\infty) = \infty$, so that $H_0^2(D_R^\infty)$ is not invariant under $C_B^{(R)}$ for $B(\infty) \neq \infty$. Thus, the operator $\Pi_-^{(R)} C_B^{(R)}$ is not generally a composition operator on $H_0^2(D_R^\infty)$ as $\Pi_-^{(R)} C_B^{(R)} f = C_B^{(R)} f - \Pi_+^{(R)} C_B^{(R)} f = C_B^{(R)} f - f(B(\infty))$, but we can relate its spectrum to the spectrum of $C_B^{(R)}$ on $H^2(D_R^\infty)$. More precisely,

$$\sigma(\Pi_-^{(R)} C_B^{(R)}) = \sigma(C_B^{(R)}) \setminus \{1\}, \tag{4.14}$$

as we shall see below. Then, using (4.14) the second assertion of the theorem follows, since

$$\begin{aligned} \sigma(\mathcal{L}') &= \sigma(C_B^{(r)}) \cup \sigma(\Pi_-^{(R)} C_B^{(R)}) \\ &= \{\lambda(z_0)^n : n \in \mathbb{N}_0\} \cup \{\lambda(\hat{z}_0)^n : n \in \mathbb{N}\} \cup \{0\}, \end{aligned}$$

and $\sigma(\mathcal{L}) = \sigma(\mathcal{L}^*) = \sigma(\mathcal{L}')$. The assertion concerning multiplicities follows from the simplicity of eigenvalues of compact composition operators.

It remains to prove (4.14). For brevity, we drop the superscript (R) from $\Pi_-^{(R)}$, $\Pi_+^{(R)}$ and $C_B^{(R)}$ since we only consider functions in $H^2(D_R^\infty)$ in what follows. Observe that for $f \in H^2(D_R^\infty)$, we have $(\Pi_+ f)(z) = f(\infty)$, which implies

$$C_B \Pi_+ = \Pi_+ \quad \text{and} \quad \Pi_- C_B = \Pi_- C_B \Pi_-. \tag{4.15}$$

Note that 1 is an eigenvalue of C_B if and only if the corresponding eigenfunction is constant. Take $\mu \in \sigma(C_B)$ with $\mu(1-\mu) \neq 0$. Since C_B is compact, there is a nonzero $f \in H^2(D_R^\infty)$ with $C_B f = \mu f$. The second equality in (4.15) now implies $\Pi_- C_B \Pi_- f = \mu \Pi_- f$. But since $\mu \neq 1$ the eigenvector f is nonconstant, so we have $0 \neq \Pi_- f \in H_0^2(D_R^\infty)$ and thus $\mu \in \sigma(\Pi_- C_B)$.

To show the converse inclusion, take $\mu \in \sigma(\Pi_- C_B)$ with $\mu \neq 0$. Since $\Pi_- C_B$ is compact, there is a nonzero $f \in H_0^2(D_R^\infty)$ with $\Pi_- C_B f = \mu f$. First observe that[3] $\mu \neq 1$. Next we note that if $\mu(\mu - 1) \neq 0$, then $(1-\mu)f - \Pi_+ C_B f \neq 0$ (for otherwise f would be zero). Finally, we use (4.15) to show that $(1-\mu)f - \Pi_+ C_B f$ is an

[3]In order to see this, note that otherwise $\Pi_- C_B f = f$, which implies $f \circ B - f = \text{const}$. However $f(B(\hat{z}_0)) - f(\hat{z}_0) = 0$, which implies $f = f \circ B$. Thus $f = \text{const}$, so $\Pi_- C_B f = 0$, contradicting the fact that $\mu \neq 0$.

eigenfunction of C_B with eigenvalue μ:

$$\begin{aligned} C_B\left((1-\mu)f - \Pi_+ C_B f\right) &= (1-\mu)(C_B f + (\mu f - \Pi_- C_B f)) - C_B \Pi_+ C_B f \\ &= \mu(1-\mu)f + (1-\mu)(I - \Pi_-)C_B f - \Pi_+ C_B f \\ &= \mu\left((1-\mu)f - \Pi_+ C_B f\right). \end{aligned}$$

Thus $\sigma(\Pi_- C_B) = \sigma(C_B) \setminus \{1\}$, as claimed. $\square$

REMARK 4.3.5. Note that $\Pi_+^{(R)} C_B^{(R)}$ is an operator of rank at most one. To see this let $e_n(z) = z^n$, then $(\Pi_+^{(R)} C_B^{(R)})(e_{-n}) = (B(\infty))^{-n} e_0$ for $n \in \mathbb{N}$. If $B(\infty) = \infty$, meaning at least one of the a_i in (4.11) is equal to 0, then $\Pi_+^{(R)} C_B^{(R)} = 0$ and $C_B^{(R)}(H_0^2(D_R^\infty)) \subseteq H_0^2(D_R^\infty)$. With slight abuse of notation, we keep writing $C_B^{(R)}$ for the restriction to $H_0^2(D_R^\infty)$, and obtain a block diagonal matrix structure of $\mathcal{L}'$ given by

$$\mathcal{L}' = \begin{pmatrix} C_B^{(r)} & 0 \\ 0 & C_B^{(R)} \end{pmatrix}. \tag{4.16}$$

Theorem 4.3.4 can now be applied to the expanding circle maps occurring in Chapter 3.

EXAMPLE 4.3.6. The simplest example is the map $B(z) = z^m$ for an integer $m \geq 2$. The proof in [**7**, Ex. 2.15] that the spectrum is $\sigma(\mathcal{L}) = \{0, 1\}$ uses exponential decay of Fourier coefficients. As B has two attracting fixed points $z_0 = 0$ and $\hat{z}_0 = \infty$ with $\lambda(z_0) = \lambda(\hat{z}_0) = 0$, this statement now simply follows from Theorem 4.3.4.

EXAMPLE 4.3.7. The family of maps $B(z) = z(\mu - z)/(1 - \overline{\mu} z)$ from (3.18) also belongs to the class of finite Blaschke products, for which the spectrum can now be deduced directly. The restriction $B|_{\mathbb{T}}$ is an expanding circle map for any $\mu \in \mathbb{D}$. The attracting fixed points are $z_0 = 0$ and $\hat{z}_0 = \infty$ with $\lambda(z_0) = \mu$ and $\lambda(\hat{z}_0) = \overline{\mu}$. Thus

$$\sigma(\mathcal{L}) = \{1\} \cup \{\mu^n : n \in \mathbb{N}\} \cup \{\overline{\mu}^n : n \in \mathbb{N}\} \cup \{0\}.$$

To conclude this section, let us mention that the expansivity condition for τ in Theorem 4.3.4 can be weakened as it is sufficient for τ to be eventually expanding. An analytic circle map $\tau\colon \mathbb{T} \to \mathbb{T}$ is called *eventually expanding* if it has an iterate that is expanding, that is, there exists an $N \in \mathbb{N}$ such that $\inf_{z \in \mathbb{T}} |(\tau^{(N)})'(z)| > 1$. If B is a Blaschke map and $\tau = B|_{\mathbb{T}}$, then $|\tau'(z)| = \sum_i (1-|a_i|^2)/|z-a_i|^2 \geq \sum_i (1-|a_i|)/(1+|a_i|)$ for $z \in \mathbb{T}$, see [**55**]. Therefore, if 0 is fixed by B, then $|\tau'| > 1$. On the other hand, any Blaschke map $\tilde{B}$ fixing $z_0 \in \mathbb{D}$ is analytically conjugate to an expanding Blaschke map B with $B(0) = 0$, via $B = \sigma \circ \tilde{B} \circ \sigma$ where σ is the Möbius map given by $\sigma(z) = \sigma^{-1}(z) = (z_0 - z)/(1 - \overline{z_0} z)$. It follows that $|(\tilde{B}^{(N)})'| > 1$ for some N. The conjugacy σ can then be used to define a transfer operator for $\tilde{B}|_{\mathbb{T}}$ on a Hardy-Hilbert space $H^2(\tilde{A})$ over a (topological) annulus $\tilde{A}$ (see [**28**, §10.1 and §10.5] for definitions) given by $\tilde{A} = \sigma(A)$ with $A \in \mathcal{A}$ a suitable annulus such that the transfer operator associated to B is well-defined and compact. By adapting arguments from Section 4.1.2 one can show its boundedness and compactness. Further, its spectrum coincides with the spectrum of $\mathcal{L}$ associated to $B|_{\mathbb{T}}$ in Theorem 4.3.4.

4.4. Application: Arbitrarily fast exponential mixing

We shall now return to the motivation of this thesis. By applying Theorem 4.3.4 to a particular family of expanding analytic maps, we obtain insight into the question of the relation between exponential mixing rates and Lyapunov exponents, considered in Chapter 2. This family is given by the finite Blaschke products of the form

$$B(z) = (-1)^m z^m \frac{(z-b)}{1-bz} \tag{4.17}$$

for $b \in (-1,1)$ and fixed integer $m \geq 2$. Then $\tau = B|_{\mathbb{T}}$ is an expanding $(m+1)$-to-1 circle map (see Figure 4.1 for $m = 2$). As the multipliers of the attracting fixed points of B are vanishing as in Example 4.3.6, we get $\sigma(\mathcal{L}) = \{0, 1\}$.

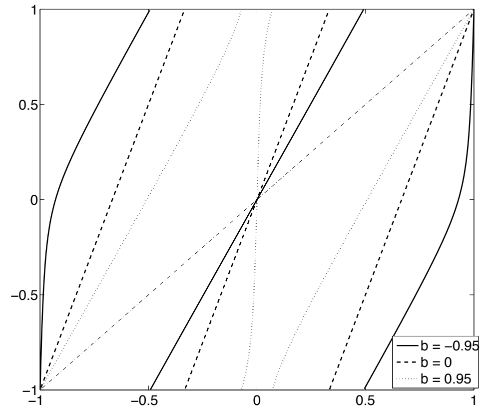

FIGURE 4.1. The circle map τ, as the restriction of B in (4.17) for $m = 2$, projected onto the interval $[-1, 1]$ for $b = -0.95, 0$ and 0.95.

It turns out that maps of the form (4.17) considered as interval maps $T\colon I \to I$ provide a class of nonlinear expanding maps for which the mixing rate α is not bounded in terms of the Lyapunov exponent Λ in the vein of Proposition 2.2.9. To show this, we shall deduce the spectrum of the transfer operator associated to T using the procedure described in Section 3.4, in particular using Lemma 3.4.2. Before doing so, we need to consider the minor technical detail, that in Lemma 3.4.2 the operator $\mathcal{L}_{\mathbb{T}}$ is considered on $H^\infty(A)$, whereas in this chapter $\mathcal{L}$ is defined on $H^2(A)$. However, the spectra of $\mathcal{L}_{\mathbb{T}}$ and $\mathcal{L}$ coincide as the next lemma shows.

LEMMA 4.4.1. *Let $\mathcal{L}$ and $\mathcal{L}_{\mathbb{T}}$ be defined as above, then $\sigma(\mathcal{L}) = \sigma(\mathcal{L}_{\mathbb{T}})$.*

PROOF. Observe that the canonical embedding $J\colon H^\infty(A) \to H^2(A)$ is continuous as $\|Jf\|_{H^2(A)} \leq \sqrt{2}$ for any $f \in H^\infty(A)$ with $\|f\|_{H^\infty(A)} = 1$. It has dense range since

$H^\infty(A)$ contains the space of Laurent polynomials, which is dense in $H^2(A)$. Then, as J intertwines $\mathcal{L}$ and $\mathcal{L}_{\mathbb{T}}$, that is $J\mathcal{L}_{\mathbb{T}} = \mathcal{L}J$, we obtain $\sigma(\mathcal{L}) = \sigma(\mathcal{L}_{\mathbb{T}})$ by [**35**] and the fact that both have discrete spectra. □

Now, to compute the multipliers of B in (4.17), note that

$$B'(z) = (-1)^m m z^{m-1} A(z) + (-1)^m z^m A'(z)\,,$$

where $A(z) = (z-b)/(1-bz)$ and $A'(z) = (1-b^2)/(1-bz)^2$. For any m, the point -1 is fixed by τ and

$$\tau'(-1) = B'(-1) = \frac{(m+1)+(m-1)b}{1+b}\,.$$

We can consider the map $T\colon I \to I$ on the interval $I = [-1,1]$ arising from τ via $p \circ T = \tau \circ p$ with a projection $p\colon I \to \mathbb{T}$ given by $p(x) = e^{i\pi x}$, which maps the interval end point -1 to the fixed point -1, and $T'(-1) = \tau'(-1)$. Let $\mathcal{L}_I\colon H^\infty(D) \to H^\infty(D)$ be the transfer operator associated to T in (3.22) with $K = m+1$. By Lemma 3.4.2 the eigenvalues of $\mathcal{L}_I$ can be divided into two classes, those that are eigenvalues of $\mathcal{L}_{\mathbb{T}}$ and those given by the inverse of the multiplier of the fixed point -1. Hence

$$\sigma(\mathcal{L}_I) = \{1,0\} \cup \left\{ \left(\frac{1+b}{(m+1)+(m-1)b} \right)^n : n \in \mathbb{N} \right\}.$$

Clearly, $(1+b)/((m+1)+(m-1)b)$ tends to 0 as $b \to -1$.

In summary, for any $m \geq 2$ one can construct analytic expanding full branch interval maps with $m+1$ branches (fixing an interval endpoint), such that the subleading eigenvalue of the associated transfer operator is arbitrarily close to zero. This answers a question raised by M. Pollicott (private communication), whether expanding interval maps can have arbitrarily small nonzero second eigenvalue.

In order to obtain Lyapunov exponents, note that by Proposition 4.3.2 any map T arising from a finite Blaschke product B (expanding on $\mathbb{T}$) has a unique ergodic acip measure μ whose density with respect to the normalised Lebesgue measure on the interval $[-1,1]$ is given by

$$\varrho(x) = \frac{1-|z_0|^2}{2|e^{i\pi x} - z_0|^2}\,,$$

where z_0 is the attracting fixed point of B in $\mathbb{D}$. The map B in (4.17) has the fixed point $z_0 = 0$, thus its acip measure is the Lebesgue measure itself. Hence, for any $b \in (-1,1)$ the Lyapunov exponent of T with respect to this invariant measure is given by $\Lambda = (1/2)\int_I \ln|T'(x)|\,dx$, or equally $(2\pi)^{-1}\int_{-\pi}^{\pi} \ln|B'(e^{i\theta})|\,d\theta$, using $B(e^{i\pi x}) = e^{i\pi T(x)}$. Interestingly, Λ can be calculated explicitly by applying Jensen's formula[4] to B'.

[4]Jensen's formula reads $\ln|f(0)| = (2\pi)^{-1}\int_{-\pi}^{\pi} \ln|f(e^{i\theta})|\,d\theta + \sum_i \ln|a_i|$, where f is holomorphic on a neighbourhood of $\overline{\mathbb{D}}$, $f(0) \neq 0$ and a_i are the zeros of f in $\mathbb{D}$, see [**89**, §3.61 and 3.62]. For a Blaschke map B with $B(0) = 0$, writing $B'(z) = z^l \cdot f(z)$ with $f(0) \neq 0$, we get $\Lambda = (2\pi)^{-1}\int_{-\pi}^{\pi} \ln|B'(e^{i\theta})|\,d\theta = \ln|f(0)| - \sum_i \ln|c_i|$ with c_i the critical points of B inside $\mathbb{D}$. If 0 is not fixed by B, then Λ can be calculated similarly using the Poisson-Jensen formula .

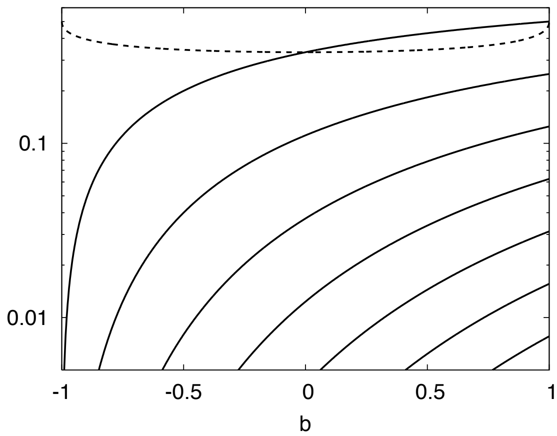

FIGURE 4.2. Eigenvalues of $\mathcal{L}_I$ for the interval map T arising from B in (4.17) with $m = 2$ for $b \in (-1, 1)$ (solid lines), in logarithmic scale. The dashed line shows $\exp(-\Lambda)$, see (4.18).

We now restrict to the case $m = 2$, where this calculation yields

$$\Lambda = \ln\left(\frac{1}{2}\left(3 + b^2 + \sqrt{(1-b^2)(9-b^2)}\right)\right), \tag{4.18}$$

see Figure 4.2.

Clearly, Λ tends to $\ln(2)$ as $b \to -1$. On the other hand, with $\lambda_2 = \lambda_2(\mathcal{L}_I)$ the second largest eigenvalue of $\mathcal{L}_I$, the mixing rate on $H^\infty(D)$ denoted by $\alpha = \alpha_{H^\infty(D)}$ as in (1.14) is given by $\alpha = -\ln \lambda_2 = -\ln((1+b)/(3+b))$. Then, α tends to infinity as $b \to -1$. Crucially, α is not bounded in terms of Λ as $b \to -1$, unlike the case of piecewise linear Markov maps in Proposition 2.2.9, where a bound $\alpha \le 2\Lambda$ has been established.

Although the Lyapunov exponent does not provide a bound for the mixing rate, interestingly its effect can be observed in the decay of a correlation function on the short time scale. Note that for b close to -1, the map T closely resembles a shifted version of the doubling map, which is the pointwise limit of T as $b \to -1$. This is reflected in the shape of the correlation function, in the sense that correlations decay exponentially with the rate $\ln(2)$ on a short time scale, prior to reaching the asymptotic rate determined by the second largest eigenvalue of $\mathcal{L}_I$.

To illustrate this, let us look numerically at the autocorrelation function for the observables $f(x) = g(x) = x$, given by

$$C_{f,g}(n) = \frac{1}{2}\int_I f(x) \cdot (g \circ T^n)(x)\, dx\,. \tag{4.19}$$

It is straightforward but slightly tedious to work out these integrals numerically to high precision[5]. The result is displayed in Figure 4.3. The asymptotic decay of

[5]Given n, one computes cylinder sets of generation n and then performs the integral over each of these intervals with a suitable integration routine. For that purpose we have used a quadruple precision version of the QUADPACK routines [**1**].

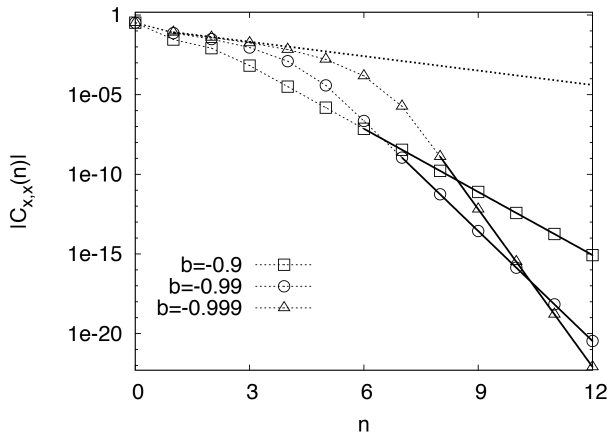

FIGURE 4.3. Autocorrelation function (4.19) of the map T arising from B in (4.17) for different values of b (symbols, with dashed lines as guide for the eye), on logarithmic scale. Solid straight lines indicate the asymptotic decay as computed from the subleading eigenvalue of $\mathcal{L}_I$. The dashed straight line displays the correlation function $C_{x,x}(n) = -1/(6 \times 2^n)$ of the pointwise limit map as $b \to -1$.

the correlation function is determined by the subleading eigenvalue $(1 + b)/(3 + b)$. However, as the parameter b approaches -1 the autocorrelation function develops a pronounced transient exponential shape which follows the correlation decay of the shifted doubling map prior to asymptotically approaching the actual rate. The transition time scale, which is relevant in applications, can be estimated by an heuristic argument, see [**84**].

While the average expansion rate in the system, quantified by the Lyapunov exponent, does not provide a bound for the mixing rate, the maximal expansion rate

$$\Lambda_+ = \lim_{n\to\infty} \sup \left\{ \frac{1}{n} \ln |(T^n)'(x)| : x \in I \right\}$$

yields[6] the bound $\alpha \leq \Lambda_+$ for any interval map T arising from an analytic expanding circle map, as the following simple corollary to Lemma 3.4.2 shows.

COROLLARY 4.4.2. *Let T, τ and their corresponding transfer operators $\mathcal{L}_{\mathbb{T}}$ and $\mathcal{L}_I$ be defined as in Lemma 3.4.2. Then,*

$$|\lambda_2| \geq e^{-\Lambda_+},$$

where $\lambda_2 = \lambda_2(\mathcal{L}_I)$ is second largest eigenvalue (in modulus) of $\mathcal{L}_I$. Equivalently, $\alpha \leq \Lambda_+$.

PROOF. Let x_0 be the interval endpoint fixed by T, then as $(T'(x_0))^{-1} \in \sigma(\mathcal{L}_I)$ we have

$$|\lambda_2| \geq |T'(x_0)|^{-1} = |(T^n)'(x_0)|^{-1/n} \geq \inf \left\{ |(T^n)'(x_0)|^{-1/n} : x \in I \right\}$$

[6]The limit exists as the sequence $(\sup\{\ln |(T^n)'(x)| : x \in I\})_{n\in\mathbb{N}}$ is subadditive.

for any $n \in \mathbb{N}$. The claim follows as $e^{-\Lambda_+} = \lim_{n\to\infty} \inf \left\{ |(T^n)'(x_0)|^{-1/n} : x \in I \right\}$. $\square$

In comparison with the bounds obtained in Chapter 2, this corollary provides a barrier for the speed of mixing in an alternative, nonlinear setting of interval maps arising from analytic expanding circle maps. While in this setup the average expansion rate Λ is replaced by the maximal expansion rate Λ_+, this observation at least partially recovers the intuition that a specific exponential mixing rate requires the presence of sufficiently strong (local) expansion.

Concluding remarks and open questions

Our work originated from the question whether the two quantifiers of chaoticity, Lyapunov exponents and mixing rates, can be related to each other. In a simple setting of topologically mixing piecewise linear expanding Markov maps on the interval, we have shown that the Lyapunov exponent provides a barrier to the exponential mixing rate, by establishing a lower bound for the subleading eigenvalue of the associated transfer operator. This bound, however, fails to generalise to a nonlinear setting, as we illustrated by a family of expanding interval maps with arbitrarily fast exponential mixing, but bounded Lyapunov exponent. Instead, the bound can be restored in spirit by replacing the Lyapunov exponent with the maximal expansion rate.

These results followed from more general considerations concerning the correlation spectrum of analytic expanding circle maps. The key step was to establish a natural representation of the adjoint of the transfer operator for these maps. For the class of finite Blaschke products (which the above family belongs to), this representation takes the form of (compact) composition operators on holomorphic function spaces, allowing for determination of the entire correlation spectrum.

Explicit knowledge of the spectrum of the transfer operator proves useful in answering certain questions and provides a testing ground for conjectures. In this final section we want to point out a number of open questions resulting from our work. We hope that some of these can be tackled with similar methods in the future.

Mixing rates and maximal expansion rates. For interval maps arising from expanding circle maps, Corollary 4.4.2 asserts the lower bound $|\lambda_2| \geq e^{-\Lambda_+}$ for the subleading eigenvalue λ_2 of the transfer operator in terms of the maximal expansion rate Λ_+. It is a natural question whether a similar bound extends to a more general class of expanding maps on the interval. It appears plausible to believe that in interval maps exponential mixing should not happen with a rate faster than with the rate given by $2\Lambda_+$. Note that for the tent map we have $|\lambda_2| = e^{-2\Lambda} = e^{-2\Lambda_+}$. Further numerical simulations for interval maps which do not arise from (smooth) circle maps motivate the following question.

Question 1. Let $T\colon I \to I$ be an analytic expanding full branch interval map, and $\mathcal{L}_I\colon H^\infty(D) \to H^\infty(D)$ the associated compact transfer operator given by (3.22) for a suitable $D \subset \mathbb{C}$ containing I. Is it then true that

$$|\lambda_2| \geq e^{-2\Lambda_+},$$

where λ_2 is the second largest eigenvalue in modulus of $\mathcal{L}_I$? Moreover, does the sharper estimate

$$|\lambda_2| \geq e^{-\Lambda_+}$$

hold, if T' has constant sign?

Nontrivial 'nonessential' spectrum. Dealing with the transfer operator on spaces of less regular functions gives rise to essential spectrum and leads to the question whether 1 may occur as the only eigenvalue outside the essential spectrum. In [**43**] the authors addressed this question by constructing an analytic expanding circle map with its transfer operator having an isolated eigenvalue different from 1 outside the essential spectrum when acting on C^1 functions. This was achieved by starting with a piecewise linear expanding circle map, for which the transfer operator considered on BV has such eigenvalue outside the essential spectrum, smoothing this map with the Gaussian convolution kernel and then applying spectral perturbation theory to the corresponding transfer operator.

It appears that the Blaschke map considered in (3.18) for $\mu = -0.7 - 0.7i$, say, provides an explicit example of analytic expanding circle map with the eigenvalues μ and $\bar{\mu}$ outside the essential spectrum on C^1. This informal statement is based on a computational bound on the essential spectral radius but probably can be made rigorous in the future.

Question 2. What are the conditions on the parameters of Blaschke maps for the existence of isolated eigenvalues outside $\rho_{ess}(\mathcal{L}|_{C^1})$? Can there be arbitrarily many such eigenvalues?

Nontrivial eigenvalues of circle maps. In [**63**, Thm. 1.2] using techniques from potential theory it was shown that there is a dense set of analytic expanding full branch interval maps which have infinitely many nontrivial eigenvalues decaying almost exponentially.

Question 3. Can one adapt these techniques together with the knowledge of the spectrum for Blaschke maps to show a similar result in the case of analytic expanding circle maps?

Rational circle maps. Theorem 4.3.4 provides the spectrum of transfer operators for rational maps τ preserving the unit circle, given by finite Blaschke products. If τ is an analytic circle map given by any rational function, it is not difficult to see that it is given by the same expression as (4.11), however with a_i in $\hat{\mathbb{C}} \setminus \mathbb{T}$ (instead of $\mathbb{D}$), leading to poles inside the unit disk, which prevent the upper triangular block structure of the adjoint operator $\mathcal{L}'$.

Question 4. Can one still use the structure of the adjoint representation $\mathcal{L}'$ to deduce some information about the spectrum?

More general transfer operators. For an expanding circle map τ arising from a finite Blaschke product, one can consider transfer operators (3.10) with more general weights w_k replacing the weights ϕ'_k. If we take $w_k = (\phi'_k)^{1-s}$ with $s \in \mathbb{N}$, then the upper triangular block structure of the adjoint $\mathcal{L}'$ in (4.9) is preserved. The upper left operator is a compact weighted composition operator on $H^2(D_r)$ given by $f \mapsto (\tau')^s \cdot (f \circ \tau)$. Its spectrum is known to consist of eigenvalues $\lambda_n = (\tau'(z_0))^{s+n}$, $n = 0, 1, \ldots$, where z_0 is the unique fixed point of τ in D_r. Clearly, these eigenvalues are in $\sigma(\mathcal{L})$. However, the lower right operator in (4.9) is not related to a compact (weighted) composition operator (compare (4.14)), as τ' is not holomorphic on D_R^∞.

QUESTION 5. Is it possible to obtain the entire spectrum of $\mathcal{L}$ with weights $w_k = (\phi'_k)^{1-s}$ for $s \in \mathbb{N}$?

'Lifts' of Blaschke products. Given an expanding circle map τ arising from a finite Blaschke product, it is possible to construct a new circle map $\tilde{\tau}$ given by the semiconjugacy $(p \circ \tilde{\tau})(z) = (\tau \circ p)(z)$ with $p(z) = z^n$ for an integer $n \geq 2$. By choosing branch cuts (arising from the n-th root of unity) appropriately, one can check that $\tilde{\tau}$ is an analytic expanding circle map, however no longer given by a Blaschke product. In particular, taking $\tau(z) = z(\mu - z)/(1 - \mu z)$ with $\mu \in (-1, 1)$ and $n = 2$, one can relate the matrix representations as in (3.19) of $\tilde{\tau}$ and τ, and deduce that the respective transfer operators $\tilde{\mathcal{L}}$ and $\mathcal{L}$ have the same nonzero eigenvalues. However, if $n > 2$ the matrix representation (3.19) as well as numerical simulations suggest that $\tilde{\mathcal{L}}$ has additional nonzero eigenvalues.

QUESTION 6. What is the spectrum of $\tilde{\mathcal{L}}$ associated to $\tilde{\tau}$ for $n > 2$?

Perturbations of Blaschke products. Let us consider a finite Blaschke product B (expanding on $\mathbb{T}$) from a complex dynamics perspective. Then $\mathbb{T}$ is the Julia set of B, and the two complementary disks in $\hat{\mathbb{C}} \setminus \mathbb{T}$ are the connected components of the Fatou set. Note that B is 'mirror symmetric' with respect to $\mathbb{T}$ in the sense that $B(z) = (\psi \circ B \circ \psi)(z)$, where $\psi(z) = 1/\bar{z}$. More generally, one can consider perturbations of Blaschke products which have a quasicircle as their Julia set (instead of $\mathbb{T}$). Similarly to the approaches taken in [**49, 50**], one can then define a transfer operator $\mathcal{L}$ on a suitable space of holomorphic functions on a neighbourhood of the Julia set and show its compactness.

QUESTION 7. In this setting it appears possible to obtain a similar structure for the adjoint operator $\mathcal{L}'$. Can the spectrum of $\mathcal{L}'$ again be deduced from certain composition operators? In this case, we would expect that the eigenvalues are not necessarily complex conjugates of each other, as the contraction rates at the fixed points in the interior of the Fatou set need not be of the same modulus, due to loss of symmetry.

Higher-dimensional dynamics. It is of great interest to determine the spectrum of transfer operators associated to higher-dimensional dynamical systems. Informally, by coupling finite Blaschke products it seems to be possible to construct higher-dimensional expanding maps, for which the correlation spectrum can be deduced from the spectrum of individual maps. The main goal is to obtain the spectrum of the transfer operator for nontrivial hyperbolic systems (see [**66**] for a study of hyperbolic diffeomorphisms of $\mathbb{T}^2$ obtained from two-dimensional Blaschke products). For this, one of the first difficulties would be to establish an appropriate function space.

QUESTION 8. Using our results on the spectrum of transfer operators for one-dimensional Blaschke products, is it possible to construct hyperbolic maps on $\mathbb{T}^2$ with explicitly known spectrum?

Appendix A: Basic spectral theory

We briefly summarise some standard results from spectral theory, see, for example, [**87**, Ch. V] for more details.

Let $L\colon V \to V$ be a bounded linear operator on a Banach space $(V, \|\cdot\|)$, and denote its kernel and image by $\ker(L)$ and $\mathrm{im}(L)$, respectively. Its *spectrum* $\sigma(L)$ is defined as the set of all $\lambda \in \mathbb{C}$ such that $(\lambda I - L)$ has no bounded inverse on V, where I is the identity operator on V. The complement of the spectrum $\mathbb{C}\setminus\sigma(L)$ is called the *resolvent set*. For each λ in the resolvent set, the *resolvent* $R(\lambda) = (\lambda I - L)^{-1}$ is well defined and holomorphic on $\mathbb{C} \setminus \sigma(L)$, in the sense that $\lambda \mapsto \phi(R(\lambda))$ is holomorphic for any bounded linear functional ϕ on the space of bounded linear operators on V.

The spectrum is a compact subset of a closed disk with radius $\rho(L) = \sup\{|z| : z \in \sigma(L)\}$. The radius $\rho(L)$ is called *spectral radius* and can be obtained as the limit

$$\rho(L) = \lim_{n\to\infty} \|L^n\|^{1/n} . \tag{A.1}$$

In particular, the latter implies that for every $\varepsilon > 0$, there is $c_\varepsilon > 0$ such that

$$\|L^n f\| \le c_\varepsilon \|f\| \left(e^\varepsilon \rho(L)\right)^n \quad \text{for all } n \in \mathbb{N}. \tag{A.2}$$

An element $\lambda \in \sigma(L)$ is an *eigenvalue* of L if $\lambda I - L$ is not injective. The *geometric multiplicity* of λ is the dimension of its eigenspace $\{v \in V : (\lambda I - L)v = 0\}$, and its *algebraic multiplicity* is the dimension of the generalised eigenspace $\{v \in V : \exists m \ge 1, (\lambda I - L)^m v = 0\}$.

The spectrum of any bounded linear operator L can be decomposed into a discrete part, made up of isolated eigenvalues of finite algebraic multiplicity, and the *essential spectrum*, denoted by $\sigma_{ess}(L)$. The *essential spectral radius* is defined as $\rho_{ess}(L) = \sup\{|z| : z \in \sigma_{ess}(L)\}$ and can be characterised as the radius $R \ge 0$ of the smallest closed disk centred at 0, such that every $\lambda \in \sigma(L)$ with $|\lambda| > R$ is an isolated eigenvalue of finite algebraic multiplicity.

A bounded operator $L\colon V \to V$ is *quasicompact* if $\rho_{ess}(L) < \rho(L)$. It is *compact* if any bounded set in V is mapped to a relatively compact set in V. The spectrum of a compact operator is a sequence of eigenvalues $(\lambda_n(L))_{n\in\mathbb{N}}$ converging to zero, together with zero itself[7].

We denote by V^* the topological dual of V, that is the space of continuous linear functionals on V. The adjoint of L is $L^*\colon V^* \to V^*$, given by the adjoint equation $l(Lv) = (L^*l)(v)$ for all $v \in V$ and $l \in V^*$. Its spectrum coincides with that of L.

[7]The set of nonzero eigenvalues may be finite, in which case we set $\lambda_n(L) = 0$ for all large $n \in \mathbb{N}$.

Appendix B: Technical proofs for Section 1.3

In this appendix we provide the ommited proofs for results stated in Section 1.3. For convenience, their statements and the standing assumptions (AS1) and (AS2) are repeated here.

(AS1) Let $T\colon X \to X$ be nonsingular with respect to m and assume that it possesses a unique acip measure μ with density $\varrho \in L^1(X,m)$ bounded away from zero and infinity, and normalised so that $\int_X \varrho \, dm = 1$.

(AS2) Let T satisfy (AS1) and let $\mathcal{L}$ be its associated transfer operator. Assume that V is an $\mathcal{L}$-invariant subspace, densely and continuously embedded in $L^1(X,m)$ such that $\mathcal{L}$ restricted to V is quasicompact.

LEMMA 1.3.4. *Let T and V satisfy* (AS2). *Then the unique invariant acip density ϱ is in V. The spectral radius of $\mathcal{L}\colon V \to V$ is 1, and if additionally the acip measure is mixing, then the only spectral point on the unit circle is 1, which is a simple eigenvalue.*

PROOF. For convenience, in this proof we write $\mathcal{L}_V$ for the transfer operator considered on V and keep writing $\mathcal{L}$ for the operator considered on $L^1(X,m)$. The two are related by

$$J\mathcal{L}_V = \mathcal{L}J,$$

where $J\colon V \to L^1(X,m)$ is the continuous embedding of V in $L^1(X,m)$. Its adjoint $J^*\colon L^1(X,m)^* \to V^*$ is injective as J has dense range. Moreover, the adjoint operators $\mathcal{L}_V^*$ and $\mathcal{L}^*$ are related by

$$\mathcal{L}_V^* J^* = J^* \mathcal{L}^*.$$

Let $l \in L^1(X,m)^*$ be given by $l(f) = \int_X f \, dm$, then from the definition of $\mathcal{L}$ in (1.6) we obtain $l(f) = l(\mathcal{L}f) = (\mathcal{L}^* l)(f)$ for any $f \in L^1(X,m)$. Thus 1 is an eigenvalue of $\mathcal{L}^*$. As J^* is injective, it follows that $1 \in \sigma(\mathcal{L}_V^*) = \sigma(\mathcal{L}_V)$. By [**35**], injectivity of J implies that every connected component of $\sigma(\mathcal{L}_V)$ has nonempty intersection with $\sigma(\mathcal{L})$. From quasicompactness of $\mathcal{L}_V$ and the fact that $\sigma(\mathcal{L})$ is contained in the unit disk, it follows that 1 is an eigenvalue of $\mathcal{L}_V$ (and the spectral radius of $\mathcal{L}_V$ is 1). For $h \neq 0$ such that $\mathcal{L}_V h = h$, we get $\mathcal{L}Jh = Jh$, which implies that Jh is a multiple of the unique invariant density ϱ as 1 is a simple eigenvalue of $\mathcal{L}$. Hence, ϱ is in V.

The assertion that the eigenvalue 1 is the only spectral point of $\mathcal{L}_V$ on the unit circle is a consequence of mixing. In order to see this, assume to the contrary that there exists $h \neq 0$ such that $\mathcal{L}_V h = \lambda h$ with $\lambda \neq 1$ but $|\lambda| = 1$. This implies that

Jh is an eigenvector of $\mathcal{L}$ with the same eigenvalue λ, which contradicts the mixing assumption.

Simplicity of the eigenvalue 1 for $\mathcal{L}_V$ follows from the same argument as for $\mathcal{L}$. $\square$

PROPOSITION 1.3.6. *Let T and V satisfy* (AS2) *with V a subspace of $L^\infty(X,m)$, and $M_\varrho(V) \subseteq V$. Additionally, assume that the unique acip measure is mixing. Suppose that $\mathcal{L}\colon V \to V$ is quasicompact. Then*

$$\alpha_V \geq -\ln \sup\{|\lambda| : \lambda \in \sigma(\mathcal{L}) \setminus \{1\}\} > 0.$$

PROOF. The proof relies on the spectral decomposition of $\mathcal{L}$, see [**87**, §V.9]. We will write I for the identity on V. By Lemma 1.3.4 the eigenvalue 1 is simple and is the only eigenvalue on $\mathbb{T}$. As $\mathcal{L}$ is quasicompact, 1 is isolated and we can define Π_1 the spectral projection of $\mathcal{L}$ associated with the eigenvalue 1, which satisfies $\Pi_1\mathcal{L} = \mathcal{L}\Pi_1 = \Pi_1$ with $\sigma((I-\Pi_1)\mathcal{L}) = \sigma(\mathcal{L}) \setminus \{1\}$, see, for example, [**87**, Thm. 9.1, Ch. V]. The projection given by $Pf = (\int_X f\,dm)\varrho$ as in (1.7) also satisfies the same relations $P\mathcal{L} = \mathcal{L}P = P$, and one can show that $P = \Pi_1$. Writing $\mathcal{L} = P + R$ with $R = (I-P)\mathcal{L}$ so that $\rho(R) = \sup\{|\lambda| : \lambda \in \sigma(\mathcal{L}) \setminus \{1\}\}$ and using the above commutation relations, we observe that $\mathcal{L}^n - P = R^n$ for $n > 0$. Now, for $f, g \in V$, from (1.9) we get

$$\begin{aligned} |C_{f,g}(n)|^{1/n} &= \left| \int_X R^n(f\varrho) \cdot g\,dm \right|^{1/n} \leq \left(\|R^n(f\varrho)\|_\infty \|g\|_\infty\right)^{1/n} \\ &\leq \left(c^2 \cdot \|R^n(f\varrho)\|_V \|g\|_V\right)^{1/n} \end{aligned}$$

with some $c > 0$, as V is continuously embedded[8] in $L^\infty(X,m)$ (that is $\|\cdot\|_\infty \leq c\,\|\cdot\|_V$). As for every $\varepsilon > 0$ we have $\|R^n(f\varrho)\|_V / \|f\varrho\|_V = O(e^{\varepsilon n}\rho(R)^n)$ as $n \to \infty$, see (A.2), it follows that

$$\limsup_{n\to\infty} |C_{f,g}(n)|^{1/n} \leq \rho(R),$$

for all $f, g \in V$. Hence, $\alpha_V \geq -\ln \rho(R)$. $\square$

LEMMA 1.3.7. *Let $L\colon V \to V$ be a compact operator on a Banach space V, with eigenvalue sequence $(\lambda_n(L))_{n\in\mathbb{N}}$, ordered by decreasing modulus, with repetitions according to algebraic multiplicity. Let V^* denote the topological dual of V. Then, for $n \in \mathbb{N}$,*

$$|\lambda_n(L)| = \inf_{\substack{W_n \subseteq V \\ \operatorname{codim} W_n < n}} \sup \left\{ \limsup_{k\to\infty} |g(L^k f)|^{1/k} : f \in W_n, g \in V^* \right\}.$$

PROOF. We will first deal with a simple case of the statement, when L has (countably many) simple eigenvalues $|\lambda_1| > |\lambda_2| > \ldots$.

For $n \in \mathbb{N}$ let $\Pi_1, \ldots, \Pi_n$ be the spectral projections associated with the eigenvalues λ_i satisfying $\Pi_i\Pi_j = 0$ for $i \neq j$ and $\Pi_i^2 = \Pi_i$, see [**87**, Thm. 9.1, Ch. V]. Moreover,

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[8]This follows by an application of the closed graph theorem and the fact that V is continuously embedded in $L^1(X,m)$.

$\Pi_i L = L\Pi_i = \lambda_i \Pi_i$ for every i. So we can write $L = \lambda_1\Pi_1 + \lambda_2\Pi_2 + \cdots + \lambda_n\Pi_n + R_n$ with $R_n \colon V \to V$ a bounded operator and $\rho(R_n) < |\lambda_n|$. We first show that

$$\inf_{\substack{W_n \subseteq V \\ \operatorname{codim} W_n < n}} \sup \left\{ \limsup_{k\to\infty} |g(L^k f)|^{1/k} : f \in W_n, g \in V^* \right\} \leq |\lambda_n|. \tag{B.1}$$

For this, observe that $W_1 = V$, and for $n > 1$ choose $W_n = \ker(\Pi_1) \cap \ker(\Pi_2) \cap \cdots \cap \ker(\Pi_{n-1})$. Note that $\operatorname{codim} W_n = n - 1$ as $\operatorname{codim} \ker(\Pi_i) = 1$ and $\Pi_i\Pi_j = \delta_{ij}\Pi_i$. Then for any $f \in W_n$, we have

$$\begin{aligned} |g(L^k f)|^{1/k} &\leq \Big(|\lambda_n|^k |g(\Pi_n f)| + |g(R_n^k f)| \Big)^{1/k} \\ &\leq |\lambda_n| \left(|g(\Pi_n f)| + c_\varepsilon \|f\|_V \|g\|_{V^*} \right)^{1/k}, \end{aligned}$$

with some $\varepsilon > 0$ and constant c_ε such that $e^\varepsilon \rho(R_n) < |\lambda_n|$ by (A.2). Thus

$$\limsup_{k\to\infty} |g(L^k f)|^{1/k} \leq |\lambda_n|,$$

for all $f \in W_n$ and $g \in V^*$, and (B.1) follows.

For the converse direction, we take any $W_n \subseteq V$ with $\operatorname{codim}(W_n) < n$, and verify the following claim:

$$\text{There exist } f \in W_n \text{ and } g \in V^* \text{ such that } g(\Pi_i f) \neq 0 \text{ for some } i = 1, \ldots, n. \tag{B.2}$$

As W_n has $\operatorname{codim}(W_n) < n$ and $S = \operatorname{im}(\Pi_1) \oplus \operatorname{im}(\Pi_2) \oplus \cdots \oplus \operatorname{im}(\Pi_n)$ has $\dim(S) = n$, it follows that $W_n \cap S \neq \{0\}$. Then there is an $f \in W_n \cap S$ such that $\Pi_i f \neq 0$ for some i. Clearly, there is a $g \in V^*$ such that $g(\Pi_i f) \neq 0$, which proves the claim.

We take i to be the smallest index satisfying (B.2), so that $\Pi_1 f = \cdots = \Pi_{i-1} f = 0$. Writing $L^k f = \lambda_i^k \Pi_i f + \tilde{R}_n^k f$ where $\tilde{R}_n = \lambda_{i+1}\Pi_{i+1} + \cdots + \lambda_n \Pi_n + R_n$ with $\rho(\tilde{R}_n) < |\lambda_i|$, and using the reverse triangle inequality, we obtain for sufficiently large k

$$\begin{aligned} |g(L^k f)|^{1/k} &\geq \Big| |\lambda_i|^k |g(\Pi_i f)| - |g(\tilde{R}_n^k f)| \Big|^{1/k} \\ &\geq |\lambda_i| \left(|g(\Pi_i f)| - \tilde{c} \frac{(e^\varepsilon \rho(\tilde{R}_n))^k}{|\lambda_i|^k} \right)^{1/k} \end{aligned}$$

for some $\varepsilon > 0$ with $e^\varepsilon \rho(\tilde{R}_n) < |\lambda_i|$ and $\tilde{c} = c_\varepsilon \|f\|_V \|g\|_{V^*}$, see (A.2). Thus,

$$\limsup_{k\to\infty} |g(L^k f)|^{1/k} \geq |\lambda_i| \geq |\lambda_n|.$$

Since W_n was arbitrary, this completes the proof for the case of simple eigenvalues with distinct moduli.

For the general case of the proof, each Π is the spectral projection onto the generalised eigenspace of some eigenvalue λ, that is $\operatorname{im}(\Pi) = \bigcup_{p\geq 0} \ker(L - \lambda I)^p$. Then, with appropriate indexing of the eigenvalues according to their algebraic multiplicities, we can write $L = (\lambda_{i_1}\Pi_1 + N_1) + (\lambda_{i_2}\Pi_2 + N_2) + \cdots + (\lambda_{i_m}\Pi_m + N_m) + R_m$, where $L\Pi_j = \Pi_j L = \lambda_{i_j}\Pi_j + N_j$ with N_j nilpotent. With these adjustments and a few straightforward (but notationally tedious) modifications, the structure of the proof remains the same as in the simple case. □

COROLLARY 1.3.8. *Let $T\colon X \to X$ and V be as in Proposition 1.3.6. Further, assume that $M_\varrho(V) = V$ and $\mathcal{L}\colon V \to V$ is compact with eigenvalue sequence $(\lambda_n(\mathcal{L}))_{n\in\mathbb{N}}$. Then, for $n > 1$,*

$$|\lambda_n(\mathcal{L})| = \inf_{\substack{W_n \subseteq V \\ \operatorname{codim} W_n < n-1}} \sup \left\{ \limsup_{k\to\infty} |C_{f,g}(k)|^{1/k} : f \in W_n, g \in V \right\}. \tag{B.3}$$

Thus, the mixing rate on V is $\alpha_V = -\ln|\lambda_2(\mathcal{L})|$.

PROOF. As $V \subseteq L^\infty(X, m)$ we can associate to each $g \in V$ an element $l_g \in V^*$ defined as $l_g(f) = \int_X fg\,dm$. Observe that

$$C_{f,g}(k) = l_g\Big((\mathcal{L}(I-P))^k(f\varrho)\Big).$$

For the equality (B.3) to hold, it is clear that the proof of Lemma 1.3.7 applies, if we can verify (B.2). Namely, by the same argument, there is $\tilde{f} \in W_n$ such that $\Pi_i(\tilde{f}) \neq 0$ for some $i = 1, \ldots, n$. As $M_\varrho(V) = V$, there is $f \in V$ such that $\tilde{f} = f\varrho$. Choosing $g = \Pi_i(f\varrho)$, the claim follows. □

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